

TUM-HEP-795/11

February 2011

Climbing NLO and NNLO Summits of Weak Decays*

Andrzej J. Buras

Physik-Department, Technische Universität München, D-85748 Garching, Germany

TUM Institute of Advanced Study, Lichtenbergstr. 2a, D-85748 Garching, Germany

Abstract

I describe the history of the calculations of NLO and NNLO QCD corrections to weak decays of mesons and particle-antiparticle mixing in the period 1988–2010. Also existing calculations of electroweak and QED corrections to these processes are included in this presentation. These efforts bear some analogies to the climbing of Himalayas and various expeditions by several teams of strongly motivated “climbers”, acting mainly in Germany, Italy, Poland and Switzerland, allowed to move this field from the LO through the NLO to the NNLO level. The material is meant to be a guide to the rich literature on NLO and NNLO corrections in question and includes several anecdotes related to the climbs that I was involved in. While, by no means this presentation should be regarded as a review of the size of NLO and NNLO corrections to weak decays, I hope that some of the comments made in the course of the presentation could turn out to be not only amusing but also instructive.

*Dedicated to the members of the Munich NLO Club in Table 1.

Contents

1	Introduction	1
2	Theoretical Framework for Weak Decays	6
2.1	OPE and Renormalization Group	6
2.2	Local Operators in the SM	8
2.2.1	Nonleptonic Operators	9
2.2.2	Dipole Operators	10
2.2.3	$\Delta F = 2$ Operators	10
2.2.4	Semileptonic Operators	11
2.3	Local Operators in Extensions of the SM	11
2.3.1	$\Delta F = 2$ Non-leptonic Operators	11
2.3.2	$\Delta F = 1$ Non-leptonic Current-Current Operators	12
2.3.3	$\Delta F = 1$ Non-leptonic Penguin Operators	12
2.3.4	Dipole Operators	13
2.3.5	$\Delta F = 1$ Semi-leptonic Operators	13
2.4	Wilson Coefficients	13
2.4.1	General Structure	13
2.4.2	Renormalization Scheme Dependence	16
2.5	Inclusive Decays	17
2.6	The Structure and the Status of the NLO and NNLO Corrections	18
2.6.1	General Comments	18
2.6.2	Different Classes of QCD Corrections	18
2.6.3	Two-Loop Anomalous Dimensions Beyond the SM	23
2.6.4	Two-Loop Electroweak Corrections	23
2.6.5	NLO QCD Calculations Beyond the SM	23
2.6.6	Penguin-Box Expansion	23
3	$\Delta S = 1$ and $\Delta B = 1$ Non-Leptonic and Semi-Leptonic Decays	26
3.1	Effective Hamiltonians	26
3.2	Current-Current Operators	27
3.3	QCD Penguin Operators	31
3.4	Electroweak Penguin Operators	36
3.5	More Operators	40
3.6	QCD Corrections to Semi-leptonic B Decays	41
3.6.1	NLO Corrections to Inclusive $B \rightarrow X_c l \nu$ Decays	42
3.6.2	NNLO Corrections to Inclusive $B \rightarrow X_c l \nu$ Decays	42
3.6.3	NLO and NNLO Corrections to Inclusive $B \rightarrow X_u l \nu_l$ Decays	42

4	$\Delta S = 2$, $\Delta B = 2$ and $\Delta B = 0$ Transitions	42
4.1	Effective Hamiltonians for $\Delta F = 2$ Transitions	42
4.2	The Top Quark Contributions	43
4.3	Charm and Top-Charm Contributions	46
4.4	$\Delta\Gamma_s$ and $\Delta\Gamma_d$ at NLO	47
4.5	NLO QCD Corrections to Tree-Level $\Delta F = 2$ Processes	47
5	Rare K and B Decays	48
5.1	Effective Hamiltonians for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$	48
5.2	Effective Hamiltonians for $K_L \rightarrow \mu^+ \mu^-$ and $B_{s,d} \rightarrow \mu^+ \mu^-$	49
5.3	NLO and NNLO QCD Calculations	50
5.4	Two-loop Electroweak Contributions	54
6	The $B \rightarrow X_s \gamma$ Decay: The K2 of Weak Decays	55
6.1	Preliminaries	55
6.2	LO Efforts	56
6.3	NLO Efforts	58
6.3.1	The Basic Structure	58
6.3.2	Wilson Coefficients of $Q_{7\gamma}$ and Q_{8g} at $\mu = \mathcal{O}(M_W)$	58
6.3.3	Anomalous Dimension Matrix	59
6.3.4	Matrix Elements	61
6.4	$B \rightarrow X_s \gamma$ at NNLO	62
6.5	$B \rightarrow X_d \gamma$	65
6.6	Exclusive Radiative Decays	65
7	$K_L \rightarrow \pi^0 l^+ l^-$ at NLO	66
7.1	Effective Hamiltonian	66
7.2	The Structure of NLO Corrections	67
8	$B \rightarrow X_s l^+ l^-$ at NLO and NNLO	68
8.1	Effective Hamiltonian	68
8.2	$B \rightarrow X_s l^+ l^-$ at NLO	69
8.3	$B \rightarrow X_s l^+ l^-$ at NNLO	69
8.4	$B \rightarrow K l^+ l^-$ and $B \rightarrow K^*(\rho) l^+ l^-$	71
9	QCD Factorization for Exclusive B Decays (by G. Buchalla)	72
9.1	Introduction	72
9.2	Factorization formula	72
9.3	NLO calculations	75
9.4	NNLO calculations	77

9.5 Outlook	78
10 Summary	79

1 Introduction

In April 1988 the workshop on “Hadronic Matrix Elements and Weak Decays” took place in the Ringberg castle at the Tegernsee lake near Munich. This workshop, organized by Jean-Marc Gérard and myself, with a great help of Willy Huber, the last secretary of Werner Heisenberg, gathered a large portion of experts that were making the first steps towards the calculation of hadronic matrix elements relevant for $K^0 - \bar{K}^0$ mixing, $B^0 - \bar{B}^0$ mixing and non-leptonic decays of K mesons, in particular $K \rightarrow \pi\pi$. Also non-perturbative aspects of semi-leptonic K -meson decays belonged to the important topics of this workshop. The representatives of lattice calculations, large N approach, QCD sum rules, hadronic sum rules and chiral perturbation theory were presenting their views on the subject. In particular Bill Bardeen summarized the large N approach to weak non-leptonic K meson decays developed by him, Jean-Marc Gérard and myself in 1986-1988 [1, 2, 3, 4]. Using this approach we were able to obtain the first quantitative, even if approximate, results in QCD for the matrix elements relevant for the $\Delta I = 1/2$ rule, ε'/ε and $K^0 - \bar{K}^0$ mixing (\hat{B}_K). These results were certainly not appreciated by other groups during this workshop that were confident to obtain much better results in the following years. In particular Luciano Maiani and Guido Martinelli promised us to provide in a few years a much better explanation of the observed $\Delta I = 1/2$ rule in $K \rightarrow \pi\pi$ decays within the lattice framework as well as the value of \hat{B}_K in QCD. Also representatives of chiral perturbation theory and hadronic sum rules were rather critical about our work. But in 1988 only very few understood our approach, the whole field was in its infancy and it is not surprising that all competing groups had rather differing views on the subject.

The workshop was certainly a great success with hot discussions in essentially all rooms of the castle and several participants not leaving it for the five days of the workshop. In spite of this it was rather clear to me that I do not want to take part in this enterprise any longer. I was sceptical that one could improve the accuracy of our calculations and I certainly did not want to be a member of a big lattice group or joining QCD sum rule and chiral perturbation groups that were active already for many years.

During the last supper of the Ringberg workshop Guido Martinelli and me realized that it would be important to calculate NLO QCD corrections to the Wilson coefficients of penguin operators relevant for $K \rightarrow \pi\pi$ decays. In 1981 Guido took part in the pioneering calculation of the two loop anomalous dimensions of the current-current operators. This calculation done in collaboration with Guido Altarelli, Curci and Petrarca [5, 6] has been unfortunately performed in the dimensional reduction scheme (DRED) [7] that was not familiar to most phenomenologists and its complicated structure discussed in detail by these authors most probably scared many from checking their results. Moreover it was known that the treatment of γ_5 in the DRED scheme, similarly to the dimensional regularization scheme with anticommuting γ_5 (known presently as the NDR scheme), may lead to mathematically inconsistent results. Consequently it was not clear in 1988 whether the result of Altarelli et al. was really correct. Being a member

of the theory group of the Max-Planck Institute for Physics in Munich (MPI) for six years I was exposed very much to this problematic. Peter Breitenlohner and Dieter Maison [8] were rather critical about the DRED and NDR schemes. According to them and other field theorists only the 't Hooft-Veltman (HV) scheme [9] for γ_5 was mathematically self-consistent. However, this scheme was also not familiar before 1988 to many phenomenologists.

While visiting Technical University of Athens in 1984 I learned about the second two-loop calculation of anomalous dimensions of current-current operators. Two young greek physicists, Tracas and Vlachos [10], performed in 1982 the Altarelli et al. calculation of 1981 in the NDR scheme, obtaining the result that differed considerably from the one of the Italian group. They could not clarify the reason for this discrepancy and in 1984 being involved heavily in other projects I simply did not have time to have a closer look at this problem.

At this last supper of the Ringberg 1988 workshop Guido told me that he will put some of his PhD students to look into NLO QCD corrections to Wilson coefficients of QCD penguin operators relevant for $K \rightarrow \pi\pi$ decays and I told him that I will look at this problem as well. However, in April 1988 I was still at the MPI and did not have any PhD students who could join me in this enterprise. Moreover, due to heavy involvement in the organization of the ICHEP 1988 in Munich and other time consuming matters like the proceedings of the 1988 Ringberg workshop [11], lectures on our large N approach to weak decays at the Summer School in Jaca (Spain) [12] and most importantly because of my moving from MPI to the Technical University Munich, I did not have time to start this new project until October of the same year.

During my summer vacation 1988 I read several books about the Himalaya expeditions. Among them the one by colonel Hunt, in which he described in detail the well known 1953 Mount Everest expedition that he organized. From these books I learned also about the competition between Reinhold Messner and polish climber Janusz Kukuczka to conquer the fourteen highest Himalayan summits, the ones over 8000 m [†]. These were truly fantastic achievements but I wondered whether the difficulty of climbing a 8000 m high mountain by an experienced mountainer could be comparable in 1988 to the difficulty of a NLO calculation of weak decays performed by an experienced physicist like me. This comparison is not fully idiotic. After all the difficult pioneering NLO and NNLO calculations in the last 22 years required not only high technical skills but also certain planing in advance and first of all psychological and physical strength. I mean here the ability to be involved in a calculation that results sometimes in a single number but lasts at least six months and often a year or longer. The air during NLO and NNLO calculations can be very thin indeed.

These thoughts prompted me to generalize my plan for the NLO analysis of $K \rightarrow \pi\pi$ decays to all relevant K and B decays including rare, radiative and in particular CP-violating decays. In 1988 this field was, with respect to NLO QCD corrections, essentially unexplored and all NLO

[†]Messner won this competition but Kukuczka was the second to reach all these highest summits by October 1988. Unfortunately he died on October 24th, 1989 in an avalanche on the South Face of Lhotse.

summits were still waiting to be conquered. Being the first to complete all these calculations would certainly be an achievement with a lasting impact on the phenomenology of weak decays.

These were my dreams of 1988. Feeling like colonel Hunt before the Mount Everest expedition I made a list of most interesting decays and the corresponding operators. This list is given in Section 2.2 and in various Tables below. As mentioned before only the current-current operators Q_1 and Q_2 have been studied at NLO before 1988 but the status of these calculations was unclear.

The next step was to find a team of physicists with whom I would perform these calculations. To do them alone in 1988 would be a pure madness. Like Guido Martinelli I could in principle count on PhD students but in October 1988 I had none who could be put on this project and even if I was hoping to get some students at TUM soon, it was not certain that it would happen. Moreover, the knowledge of gauge theories at TUM in 1988 was very limited (Hans Peter Nilles and me were hired to change this situation) and without a series of lectures on renormalization, renormalization group methods and loop calculations, sending young students to do NLO calculations in QCD would be impractical and certainly irresponsible. I estimated that before the fall of 1989 I could not count on any help from my future PhD students and/or post-docs that were supposed to arrive in October 1989.

However, I certainly could not wait until the fall of 1989. Afterall I was convinced that Guido already worked on this project with his students. Therefore I told Jean-Marc Gérard, who was at the MPI at that time, about my plans. Between 1984 and 1988 we have written 11 papers together and I was convinced that he was the right person for this grand project. Unfortunately Jean-Marc did not want to join me in this expedition. He basically told me that I was crazy to think about calculating NLO corrections to weak decays that were polluted by hadronic uncertainties. In principle I could also ask Bill Bardeen with whom I did my first NLO calculations for deep inelastic structure functions and photon structure functions ten years earlier [13, 14]. But we were separated by the Atlantic and moreover I had some doubts that Bill would be interested in this project.

On my last day at MPI, the members of the MPI theory group were giving 5 min talks about their research, in the spirit of similar talks in the Theory Group at CERN. At this meeting I informed my MPI colleagues about my project and that I was looking for collaborators. There was no reaction. I left MPI rather frustrated.

Few days later, sitting already in my office at TUM, I got a phone call from Peter Weisz, who joined MPI few months earlier. I knew Peter from his work with Martin Lüscher, but as his field was rather different from mine I had only a few conversations with him until then. To my great surprise, Peter was very much interested in my project and asked me whether he could join me in this enterprise. I told him that I was delighted. On this day the Munich NLO Club (MNLC) was born. The club consisted only of two members but our team could start the first climb.

24 years later I can report that the MNLC consists of 31 members, most of them working now

Table 1: Members of the MNLC. My PhD students are given in **boldface** and the PhD students of my PhD students (my grandchildren) in "*slanted*". The remaining members were assistants, post docs or visitors in my group or at MPI.

Patricia Ball	<i>Matthäus Bartsch</i>	<i>Guido Bell</i>
Christoph Bobeth	<i>Stefan Bosch</i>	<i>Joachim Brod</i>
Gerhard Buchalla	Andrzej J. Buras	Kostja Chetyrkin
Andrzej Czarnecki	Thorsten Ewerth	Robert Fleischer
Paolo Gambino	<i>Jennifer Girrbach</i>	Martin Gorbahn
Ulrich Haisch	Stefan Herrlich	Matthias Jamin
Sebastian Jäger	Axel Kwiatkowski	Markus E. Lautenbacher
Alexander Lenz	Mikolaj Misiak	Manfred Münz
Ulrich Nierste	Gaby Ostermaier	<i>Volker Pilipp</i>
Nikolas Pott	Emanuel Stamou	Jörg Urban
Peter Weisz		

outside Munich. Their names are listed in Table 1[‡]. The grand project that I outlined in 1988 and started with Peter Weisz soon after turned out to be a great success. Peter was an active member of the MNLC only in the period 1988 – 1992 but these were very important years and his participation had an invaluable impact on the full project. The project has been completed within the Standard Model (SM) in the first years of this millennium. Also some calculations beyond the SM, in particular within the MSSM, have been done. We were not always the first to climb a given NLO summit, but MNLC is basically the only group that calculated NLO corrections to all relevant decays within the SM. In the last decade several NNLO calculations have also been performed in our club.

Three generations of physicists took part in this enterprise with my PhD students in Table 1 given in boldface and the PhD students of my PhD students in "*slanted*". Bartsch, Bell, Bosch and Pilipp were PhD students of Gerhard Buchalla at the Ludwig-Maximilian University in Munich and Joachim Brod graduated in Karlsruhe under the supervision of Uli Nierste, becoming the member of MNLC only two years ago as a PostDoc in the group of Martin Gorbahn. Emanuel Stamou got his Diploma under supervision of Martin Gorbahn. The remaining members except for Peter Weisz and myself were assistants, post docs or visitors in my group or at MPI.

In the days of SPIRES and INSPIRE it is easy to verify that our work had an impact on particle physics. Around 100 papers on NLO and NNLO corrections have been published by the members of the MNLC, where mainly papers are counted in which NLO and NNLO calculations have been performed and not papers in which only the phenomenological implications of these

[‡]A NLO calculation for a weak decay in Munich is necessary for a membership in the MNLC.

calculations have been analyzed. As you will see at the end of this writing, the papers from our club amount to roughly 2/3 of all papers in the field of weak decays in which such calculations have been performed. Our papers collected over 14 000 citations with 19 papers above 250 citations and 23 with 100+. Moreover the Hirsch number, taking into account that *only* roughly 90 papers have been published, is rather impressive: $h=57$. This number will certainly increase in time.

The purpose of the following presentation is the recollection of these activities and a summary of the present status of NLO and NNLO calculations in weak decays. I have organized the material in the following manner. In the next section I will summarize the theoretical framework for weak decays. This will be a compact presentation to which I will refer frequently in subsequent sections. The full exposition of the technicalities that I will try to avoid as much as possible can be found in the Rev. Mod. Phys. article written in collaboration with Gerhard Buchalla and Markus Lautenbacher [15], my Les Houches lectures [16] and of course in the original papers. Section 3 is devoted to NLO QCD corrections to $\Delta S = 1$ and $\Delta B = 1$ non-leptonic decays. I will be rather detailed about the structure of QCD corrections to these decays because the operators involved there play an important role, even if indirectly, in essentially all weak decays. Also existing NNLO calculations for these decays will be summarized. Section 4 describes $\Delta S = 2$ and $\Delta B = 2$ transitions in some detail and very briefly $\Delta B = 0$ transitions. Section 5 is devoted to rare K and B decays, in particular $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, $K_L \rightarrow \pi^0 \nu \bar{\nu}$ and $B_{s,d} \rightarrow \mu^+ \mu^-$. Section 6 is devoted to the K2 of weak decays, the inclusive decay $B \rightarrow X_s \gamma$ with a few comments on the $B \rightarrow X_s$ gluon decay. In addition to a detailed description of the history of NLO calculations we will summarize the present status of NNLO calculations. Finally, we will list the literature for the corresponding exclusive decays $B \rightarrow K^*(\rho) \gamma$. In section 7 we discuss the NLO corrections to $K_L \rightarrow \pi^0 l^+ l^-$ and in Section 8 the NLO and NNLO corections to $B \rightarrow X_s \ell^+ \ell^-$. Here also the decays $B \rightarrow K^*(\rho) l^+ l^-$ will be mentioned.

A very special Section is Section 9 because it is not written by me but by one of the prominent members of MNLC, Gerhard Buchalla, who is also one of the fathers of the QCD Factorization approach to non-leptonic decays. I thought that the QCD calculations in this approach should also have a place in this presentation and I asked Gerhard to help me in this matter. We conclude in Section 10.

While in certain parts of our review we will enter some details, the material is meant to be a guide to the rich literature on NLO and NNLO corrections to weak decays. It is certainly not a review of the subject in the spirit of [15, 16]. I just wanted to summarize what has been done during the last twenty two years, listing in particular the first climbs of the existing NLO and NNLO summits and few subsequent climbs that used different methods or routes to reach a given summit. Thus the full material can be considered as a chronicle of NLO and NNLO calculations (1988-2010) in weak decays with several anecdotes behind the scene related to the climbs that I was involved in and several, hopefully, instructive comments for non-experts that

probably are hard to find in the most recent very technical literature on NNLO corrections.

Before describing my NLO-story in more explicit terms I will make an express review of the theoretical framework for weak decays summarizing at the end the present status of NLO and NNLO calculations that are discussed in detail in the subsequent sections. I should stress that several NLO and NNLO QCD calculations in the framework of SCET, QCD sum rules, light-cone sum rules, for non-leading terms in heavy quark expansions, heavy quark effective theory, charmonia are left out from this presentation because I did not participate in these studies. With the help of SPIRES or INSPIRE clicking the names of Ball, Bauer, Beneke, Brambilla, Chetyrkin, Hoang, Jamin, Kühn, Neubert, Stewart and several other masters of these fields one can easily find all papers on these topics. The stories behind these calculations are unknown to me. In this context one should mention in particular numerous papers of Matthias Jamin on QCD corrections relevant for QCD sum rules, numerous papers by the Karlsruhe QCD club lead by Hans Kühn, in particular their results on quark masses, and the work of Andre Hoang and his collaborators among the others.

The following section is rather heavy but I hope that it will facilitate the reading of the subsequent sections for non-experts. Experts, knowing this technology, can skip this Section to go directly to Sections 3-10 in order to check whether I cited them properly. On the other hand the classification of QCD corrections into eight classes, presented in Section 2.6 could also be of interest to them.

2 Theoretical Framework for Weak Decays

2.1 OPE and Renormalization Group

The basis for any serious phenomenology of weak decays of hadrons is the *Operator Product Expansion* (OPE) [17, 18], which allows to write the effective weak Hamiltonian simply as follows

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_i V_{CKM}^i C_i(\mu) Q_i . \quad (2.1)$$

Here G_F is the Fermi constant and Q_i are the relevant local operators which govern the decays in question. As we will see below they are built out of quark and lepton fields. The Cabibbo-Kobayashi-Maskawa factors V_{CKM}^i [19, 20] and the Wilson coefficients $C_i(\mu)$ describe the strength with which a given operator enters the Hamiltonian. The latter coefficients can be considered as scale dependent “couplings” related to “vertices” Q_i and can be calculated using perturbative methods as long as the scale μ is not too small.

An amplitude for a decay of a given meson $M = K, B, \dots$ into a final state $F = \pi\nu\bar{\nu}, \pi\pi, DK$ is then simply given by

$$A(M \rightarrow F) = \langle F | \mathcal{H}_{eff} | M \rangle = \frac{G_F}{\sqrt{2}} \sum_i V_{CKM}^i C_i(\mu) \langle F | Q_i(\mu) | M \rangle, \quad (2.2)$$

where $\langle F|Q_i(\mu)|M\rangle$ are the matrix elements of Q_i between M and F , evaluated at the renormalization scale μ .

The essential virtue of the OPE is this one. It allows to separate the problem of calculating the amplitude $A(M \rightarrow F)$ into two distinct parts: the *short distance* (perturbative) calculation of the coefficients $C_i(\mu)$ and the *long-distance* (generally non-perturbative) calculation of the matrix elements $\langle Q_i(\mu)\rangle$. The scale μ separates, roughly speaking, the physics contributions into short distance contributions contained in $C_i(\mu)$ and the long distance contributions contained in $\langle Q_i(\mu)\rangle$. Our presentation is mainly devoted to the calculations of the coefficients $C_i(\mu)$ within the SM and some of its extensions. Only in the case of inclusive B decays we will also look at the evaluation of the actual decay as it can be done in perturbation theory.

It should be stressed at this point that our presentation would not exist without the asymptotic freedom in QCD [21, 22] that allows the calculations of Wilson coefficients by means of ordinary or renormalization group improved perturbation theory. The precision of these calculations increased in the last twenty years not only because of NLO and NNLO QCD calculations but also because of the more accurate determination of the strong coupling α_s for which the most recent result reads [23]:

$$\alpha_s^{\overline{\text{MS}}}(M_Z) = 0.1184 \pm 0.0007 . \quad (2.3)$$

Now, the coefficients C_i include in addition to tree level contributions from the W -exchange, virtual top quark contributions and contributions from other heavy particles such as W , Z bosons, charged Higgs particles, supersymmetric particles in the supersymmetric extensions of the SM and other heavy objects in numerous extensions of this model. Consequently $C_i(\mu)$ depend generally on m_t and also on the masses of new particles if extensions of the SM are considered. This dependence can be found by evaluating so-called *box* and *penguin* diagrams with full W , Z , top quark and new particles exchanges and *properly* including short distance QCD effects. The latter govern the μ -dependence of $C_i(\mu)$. In models in which GIM mechanism [24] is absent, also *tree* diagrams can contribute to flavour changing neutral current (FCNC) processes. The point is that a given C_i receives generally contributions from all these three classes of diagrams.

The value of μ can be chosen arbitrarily but the final result must be μ -independent. Therefore the μ -dependence of $C_i(\mu)$ has to cancel the μ -dependence of $\langle Q_i(\mu)\rangle$. In other words as far as heavy mass independent terms are concerned it is a matter of choice what exactly belongs to $C_i(\mu)$ and what to $\langle Q_i(\mu)\rangle$. This cancellation of the μ -dependence involves generally several terms in the expansion in (2.2). The coefficients $C_i(\mu)$ depend also on the renormalization scheme. This scheme dependence must also be canceled by the one of $\langle Q_i(\mu)\rangle$ so that the physical amplitudes are renormalization scheme independent. Again, as in the case of the μ -dependence, the cancellation of the renormalization scheme dependence involves generally several terms in the expansion (2.2). One of the type of scheme dependences is the manner in which

γ_5 is defined in $D = 4 - 2\epsilon$ dimensions implying for instance the three schemes NDR, HV and DRED mentioned earlier.

Although μ is in principle arbitrary, it is customary to choose μ to be of the order of the mass of the decaying hadron. This is $\mathcal{O}(m_b)$ and $\mathcal{O}(m_c)$ for B decays and D decays respectively. In the case of K decays the typical choice is $\mu = \mathcal{O}(1 - 2 \text{ GeV})$ instead of $\mathcal{O}(m_K)$, which is much too low for any perturbative calculation of the couplings C_i . Now due to the fact that $\mu \ll M_{W,Z}$, m_t , large logarithms $\ln M_W/\mu$ compensate in the evaluation of $C_i(\mu)$ the smallness of the QCD coupling constant α_s and terms $\alpha_s^n (\ln M_W/\mu)^n$, $\alpha_s^n (\ln M_W/\mu)^{n-1}$ etc. have to be resummed to all orders in α_s before a reliable result for C_i can be obtained. This can be done very efficiently by means of the renormalization group methods. The resulting *renormalization group improved* perturbative expansion for $C_i(\mu)$ in terms of the effective coupling constant $\alpha_s(\mu)$ does not involve large logarithms and is more reliable. The related technical issues are discussed in detail in [15] and [16] and we will recall here only those that are essential for our presentation.

All this looks rather formal but in fact should be familiar. Indeed, in the simplest case of the β -decay, \mathcal{H}_{eff} takes the familiar form

$$\mathcal{H}_{eff}^{(\beta)} = \frac{G_F}{\sqrt{2}} \cos \theta_c [\bar{u} \gamma_\mu (1 - \gamma_5) d \otimes \bar{e} \gamma^\mu (1 - \gamma_5) \nu_e] , \quad (2.4)$$

where V_{ud} has been expressed in terms of the Cabibbo angle [19]. In this particular case the Wilson coefficient is equal unity and the local operator, the object between the square brackets, is given by a product of two $V - A$ currents. Equation (2.4) represents the Fermi theory for β -decays as formulated by Sudarshan and Marshak [25] and Feynman and Gell-Mann [26] more than fifty years ago, except that in (2.4) the quark language has been used and following Cabibbo a small departure of V_{ud} from unity has been incorporated. In this context the basic formula (2.1) can be regarded as a generalization of the Fermi Theory to include all known quarks and leptons as well as their strong and electroweak interactions as summarized by the SM.

Due to the interplay of electroweak and strong interactions the structure of the local operators is much richer than in the case of the β -decay. They can be classified with respect to Lorentz structure, Dirac structure, the colour structure and the type of quarks and leptons relevant for a given decay. We will now list all the operators whose Wilson coefficients will be mentioned in subsequent sections.

2.2 Local Operators in the SM

We give below first a list of operators that play the role in weak B decays. Typical diagrams in the full theory from which these operators originate are shown in Fig. 1. The cross in the diagram 1d indicates that magnetic penguins originate from the mass-term on the external line in the usual QCD or QED penguin diagrams. The operators relevant for K decays are discussed subsequently.

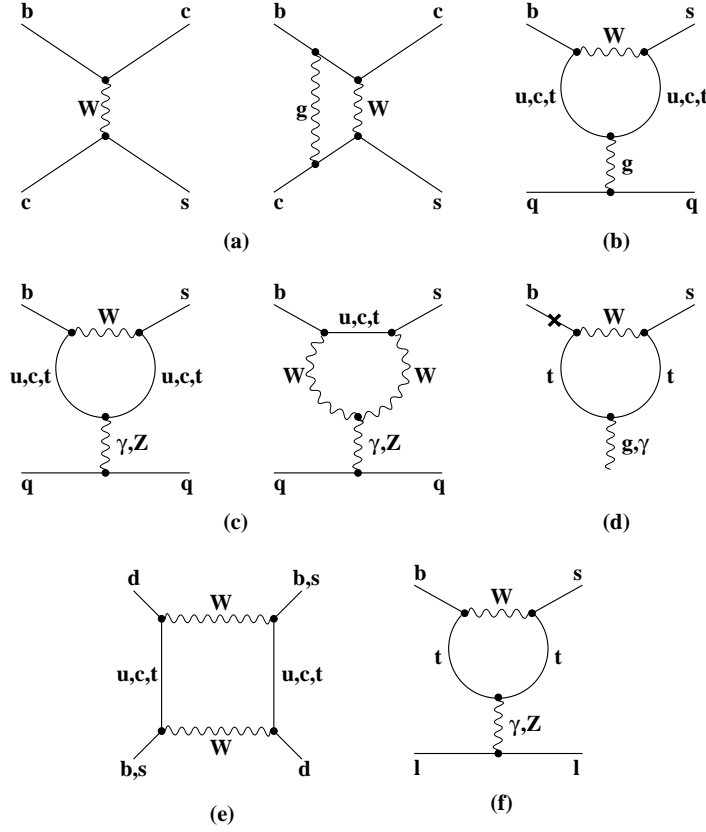


Figure 1: Typical Tree, Penguin and Box Diagrams in the SM.

2.2.1 Nonleptonic Operators

Of particular interest are the operators involving quarks only. In the case of the $\Delta B = 1$ transitions the relevant set of operators is given as follows:

Current–Current (Fig. 1a):

$$Q_1 = (\bar{c}_\alpha b_\beta)_{V-A} (\bar{s}_\beta c_\alpha)_{V-A} \quad Q_2 = (\bar{c}b)_{V-A} (\bar{s}c)_{V-A} \quad (2.5)$$

QCD–Penguins (Fig. 1b):

$$Q_3 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V-A} \quad Q_4 = (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_\beta q_\alpha)_{V-A} \quad (2.6)$$

$$Q_5 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V+A} \quad Q_6 = (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_\beta q_\alpha)_{V+A} \quad (2.7)$$

Electroweak Penguins (Fig. 1c):

$$Q_7 = \frac{3}{2} (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}q)_{V+A} \quad Q_8 = \frac{3}{2} (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}_\beta q_\alpha)_{V+A} \quad (2.8)$$

$$Q_9 = \frac{3}{2} (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}q)_{V-A} \quad Q_{10} = \frac{3}{2} (\bar{s}_\alpha b_\beta)_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}_\beta q_\alpha)_{V-A} \quad (2.9)$$

Here, α, β denote colours and e_q denotes the electric quark charges reflecting the electroweak origin of Q_7, \dots, Q_{10} . Finally, $(\bar{c}b)_{V-A} \equiv \bar{c}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha$.

These operators play a crucial role in non-leptonic decay of B_s and B_d mesons and have through mixing under renormalization also an impact on other processes as we will see below. In this context let me also make one useful remark. In the literature operators in B physics appear sometimes with $(\bar{b}s)_{V-A}$ or $(\bar{s}b)_{V-A}$, dependently whether B_s^0 or \bar{B}_s^0 are studied, respectively. When using their Wilson coefficients given in the literature it is crucial to remember that they are complex conjugates of each other. This distinction is crucial for obtaining correct CP asymmetries.

For non-leptonic K decays the flavours have to be changed appropriately. Explicit expressions can be found in [15, 16]. In particular the analogs of Q_1 and Q_2 govern the $\Delta I = 1/2$ rule in $K_L \rightarrow \pi\pi$ decays, while the corresponding QCD penguins and electroweak penguins enter directly the ratio ε'/ε .

2.2.2 Dipole Operators

In the case of $B \rightarrow X_s \gamma$ decay and $B \rightarrow X_s l^+ l^-$ decays as well as corresponding exclusive decays the crucial role is played by

Magnetic Penguins (Fig. 1d):

$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1 + \gamma_5) b_\alpha F_{\mu\nu} \quad Q_{8G} = \frac{g}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1 + \gamma_5) T_{\alpha\beta}^a b_\beta G_{\mu\nu}^a \quad (2.10)$$

Again, when using the results in the literature care must be taken whether b or \bar{b} is present in the operator and what are the factors multiplying the Dirac structures. The operator Q_{8G} can also be relevant in nonleptonic decays. Also magnetic penguins with $(1 + \gamma_5)$ replaced by $(1 - \gamma_5)$ are present but they are suppressed within the SM with respect to the operators in (2.10) by m_s/m_b .

2.2.3 $\Delta F = 2$ Operators

In the case of $K^0 - \bar{K}^0$ mixing and $B_d^0 - \bar{B}_d^0$ mixing the relevant operators within the SM are

$\Delta S = 2$ and $\Delta B = 2$ Operators (Fig. 1e):

$$Q(\Delta S = 2) = (\bar{s}d)_{V-A} (\bar{s}d)_{V-A} \quad Q(\Delta B = 2) = (\bar{b}d)_{V-A} (\bar{b}d)_{V-A} \quad (2.11)$$

For $B_s^0 - \bar{B}_s^0$ mixing one has to replace d by s in the last operator.

2.2.4 Semileptonic Operators

In the case of $B \rightarrow X_s l^+ l^-$ also the following operators originating in **Fig. 1f** on top of magnetic penguins contribute

$$Q_{9V} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_V \quad Q_{10A} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_A. \quad (2.12)$$

Changing appropriately flavours one obtains the corresponding operators relevant for $B \rightarrow X_d l^+ l^-$ and $K_L \rightarrow \pi^0 l^+ l^-$

The rare decays $B \rightarrow X_s \nu \bar{\nu}$, $B \rightarrow K^* \nu \bar{\nu}$, $B \rightarrow K \nu \bar{\nu}$ and $B_s \rightarrow \bar{\mu}\mu$ are governed by

$$Q_{\nu\bar{\nu}}(B) = (\bar{s}b)_{V-A}(\bar{\nu}\nu)_{V-A} \quad Q_{\mu\bar{\mu}}(B) = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_{V-A}. \quad (2.13)$$

The rare decays $K \rightarrow \pi \nu \bar{\nu}$ and $K_L \rightarrow \bar{\mu}\mu$ are governed on the other hand by

$$Q_{\nu\bar{\nu}}(K) = (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A} \quad Q_{\mu\bar{\mu}}(K) = (\bar{s}d)_{V-A}(\bar{\mu}\mu)_{V-A}. \quad (2.14)$$

2.3 Local Operators in Extensions of the SM

New physics (NP) can generate new operators. Typically new operators are generated through the presence of right-handed (RH) currents and *scalar* currents with the latter strongly suppressed within the SM. New gauge bosons and scalar exchanges are at the origin of these operators that can have important impact on phenomenology. Below we give examples of new operators being aware that this list is incomplete. Much more extensive discussion can be found in [27, 28].

2.3.1 $\Delta F = 2$ Non-leptonic Operators

For definiteness, we shall consider here operators responsible for the $K^0-\bar{K}^0$ mixing. There are 8 such operators of dimension 6. They can be split into 5 separate sectors, according to the chirality of the quark fields they contain. The operators belonging to the first three sectors (VLL, LR and SLL) read [29] (our competition in Rome [30] uses a different basis):

$$\begin{aligned} Q_1^{\text{VLL}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{s}^\beta \gamma^\mu P_L d^\beta), \\ Q_1^{\text{LR}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{s}^\beta \gamma^\mu P_R d^\beta), \\ Q_2^{\text{LR}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_R d^\beta), \\ Q_1^{\text{SLL}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_L d^\beta), \\ Q_2^{\text{SLL}} &= (\bar{s}^\alpha \sigma_{\mu\nu} P_L d^\alpha)(\bar{s}^\beta \sigma^{\mu\nu} P_L d^\beta), \end{aligned} \quad (2.15)$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ and $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$. The operators belonging to the two remaining sectors (VRR and SRR) are obtained from Q_1^{VLL} and Q_i^{SLL} by interchanging P_L and P_R . In the SM only the operator $Q_1^{\text{VLL}} = Q(\Delta S = 2)/4$ is present.

2.3.2 $\Delta F = 1$ Non-leptonic Current-Current Operators

In the present section, we list the current–current four-quark $\Delta F = 1$ operators. For this purpose, we choose the operators in such a manner that all the four flavours they contain are different: \bar{s} , d , \bar{u} , c . In such a case, the only possible diagrams are the current–current ones. Penguin diagrams are discussed subsequently.

Twenty linearly independent operators can be built out of four different quark fields. They can be split into 8 separate sectors, between which there is no mixing. The operators belonging to the first four sectors (VLL, VLR, SLR and SLL) read [29]

$$\begin{aligned}
Q_1^{\text{VLL}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\beta)(\bar{u}^\beta \gamma^\mu P_L c^\alpha) = \tilde{Q}_{V_L V_L}, \\
Q_2^{\text{VLL}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{u}^\beta \gamma^\mu P_L c^\beta) = Q_{V_L V_L}, \\
Q_1^{\text{VLR}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\beta)(\bar{u}^\beta \gamma^\mu P_R c^\alpha) = \tilde{Q}_{V_L V_R}, \\
Q_2^{\text{VLR}} &= (\bar{s}^\alpha \gamma_\mu P_L d^\alpha)(\bar{u}^\beta \gamma^\mu P_R c^\beta) = Q_{V_L V_R}, \\
Q_1^{\text{SLR}} &= (\bar{s}^\alpha P_L d^\beta)(\bar{u}^\beta P_R c^\alpha) = \tilde{Q}_{LR}, \\
Q_2^{\text{SLR}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{u}^\beta P_R c^\beta) = Q_{LR}, \\
Q_1^{\text{SLL}} &= (\bar{s}^\alpha P_L d^\beta)(\bar{u}^\beta P_L c^\alpha) = \tilde{Q}_{LL}, \\
Q_2^{\text{SLL}} &= (\bar{s}^\alpha P_L d^\alpha)(\bar{u}^\beta P_L c^\beta) = Q_{LL}, \\
Q_3^{\text{SLL}} &= (\bar{s}^\alpha \sigma_{\mu\nu} P_L d^\beta)(\bar{u}^\beta \sigma^{\mu\nu} P_L c^\alpha) = \tilde{Q}_{T_L T_L}, \\
Q_4^{\text{SLL}} &= (\bar{s}^\alpha \sigma_{\mu\nu} P_L d^\alpha)(\bar{u}^\beta \sigma^{\mu\nu} P_L c^\beta) = Q_{T_L T_L},
\end{aligned} \tag{2.16}$$

where on the r.h.s. we have shown the notation of the Rome group [30].

The operators belonging to the four remaining sectors (VRR, VRL, SRL and SRR) are obtained from the above by interchanging P_L and P_R . Obviously, it is sufficient to calculate the anomalous dimensions (ADMs) only for the VLL, VLR, SLR and SLL sectors. The “mirror” operators in the VRR, VRL, SRL and SRR sectors will have exactly the same properties under QCD renormalization. On the other hand their Wilson coefficients being governed by weak interactions can be different. In the SM only the operators Q_1^{VLL} and Q_2^{VLL} are present.

2.3.3 $\Delta F = 1$ Non-leptonic Penguin Operators

The operators in (2.15) and (2.16) do not constitute the full set of six-dimensional four quark operators contributing to $\Delta F = 1$ processes. In addition to QCD penguins and electroweak penguins of the SM there are other penguin operators. In our paper [29] we have therefore generalized our analysis of two-loop anomalous dimensions to the full set of $\Delta F = 1$ four-quark operators. These results are much less known but should be useful in the extensions of the SM one day. The list of these operators can be found in [29].

2.3.4 Dipole Operators

In the presence of right-handed (RH) currents, mediated for instance by a very heavy W_R in left-right symmetric models the magnetic penguins

$$\tilde{Q}_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1 - \gamma_5) b_\alpha F_{\mu\nu} \quad \tilde{Q}_{8G} = \frac{g}{8\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} (1 - \gamma_5) T_{\alpha\beta}^a b_\beta G_{\mu\nu}^a \quad (2.17)$$

could be important.

2.3.5 $\Delta F = 1$ Semi-leptonic Operators

Concerning the semi-leptonic operators in the extensions of the SM the typical examples of operators related to the presence of RH *currents* are

$$\tilde{Q}_{9V} = (\bar{s}b)_{V+A}(\bar{\mu}\mu)_V \quad \tilde{Q}_{10A} = (\bar{s}b)_{V+A}(\bar{\mu}\mu)_A. \quad (2.18)$$

$$\tilde{Q}_{\nu\bar{\nu}}(B) = (\bar{s}b)_{V+A}(\bar{\nu}\nu)_{V-A} \quad \tilde{Q}_{\mu\bar{\mu}}(B) = (\bar{s}b)_{V+A}(\bar{\mu}\mu)_{V-A}. \quad (2.19)$$

$$\tilde{Q}_{\nu\bar{\nu}}(K) = (\bar{s}d)_{V+A}(\bar{\nu}\nu)_{V-A} \quad \tilde{Q}_{\mu\bar{\mu}}(K) = (\bar{s}d)_{V+A}(\bar{\mu}\mu)_{V-A}. \quad (2.20)$$

If *scalar currents* resulting from scalar exchanges like the heavy Higgs in the 2HDM models or sparticles in the MSSM are present, scalar operators enter the game. The most prominent are the ones that govern the $B_s \rightarrow \mu^+ \mu^-$ decay in 2HDMs and the MSSM at large $\tan\beta$:

$$Q_S = (\bar{s}P_L b)(\bar{\mu}\mu) \quad Q_P = (\bar{s}P_L b)(\bar{\mu}\gamma_5\mu). \quad (2.21)$$

$$\tilde{Q}_S = (\bar{s}P_R b)(\bar{\mu}\mu) \quad \tilde{Q}_P = (\bar{s}P_R b)(\bar{\mu}\gamma_5\mu). \quad (2.22)$$

2.4 Wilson Coefficients

2.4.1 General Structure

The main objects of interest in this review are the QCD and electroweak corrections to the Wilson coefficients of the operators listed above. Once these coefficients have been calculated at a high energy scale like M_W [§], the renormalization group methods allow to calculate them at low energy scales at which the matrix elements are evaluated by means of non-perturbative methods. Denoting this lower scale simply by μ the general expression for $C_i(\mu)$ is given by:

$$\vec{C}(\mu) = \hat{U}(\mu, M_W) \vec{C}(M_W), \quad (2.23)$$

[§] $C_i(M_W)$ are often called *matching conditions* as they are found through matching of the full theory with heavy fields as dynamical degrees of freedom to the effective theory where only light fields are dynamical.

where \vec{C} is a column vector built out of C_i 's. $\vec{C}(M_W)$ are the initial conditions which depend on the short distance physics at high energy scales. In particular they depend on m_t and the masses and couplings of new heavy particles in the extensions of the SM. We set the high energy scale at M_W , but other choices are clearly possible. $\hat{U}(\mu, M_W)$, the evolution matrix from M_W down to μ , is given as follows:

$$\hat{U}(\mu, M_W) = T_g \exp \left[\int_{g(M_W)}^{g(\mu)} dg' \frac{\hat{\gamma}^T(g')}{\beta(g')} \right] \quad (2.24)$$

with g denoting the QCD effective coupling constant and T_g an ordering operation defined in [16]. $\beta(g)$ governs the evolution of g and $\hat{\gamma}$ is the anomalous dimension matrix of the operators involved. The structure of this equation makes it clear that the renormalization group approach goes beyond the usual perturbation theory. Indeed $\hat{U}(\mu, M_W)$ sums automatically large logarithms $\log M_W/\mu$ which appear for $\mu \ll M_W$. In the so-called leading logarithmic approximation (LO) terms $(g^2 \log M_W/\mu)^n$ are summed. The next-to-leading logarithmic correction (NLO) to this result involves summation of terms $(g^2)^n (\log M_W/\mu)^{n-1}$ and so on. This hierarchic structure gives the renormalization group improved perturbation theory.

As an example let us consider only QCD effects and the case of a single operator so that (2.23) reduces to

$$C(\mu) = U(\mu, M_W) C(M_W) \quad (2.25)$$

with $C(\mu)$ denoting the coefficient of the operator in question.

Keeping the first three terms in the expansions of $\gamma(g)$ and $\beta(g)$ in powers of g :

$$\gamma(g) = \gamma^{(0)} \frac{\alpha_s}{4\pi} + \gamma^{(1)} \left(\frac{\alpha_s}{4\pi} \right)^2 + \gamma^{(2)} \left(\frac{\alpha_s}{4\pi} \right)^3, \quad \alpha_s = \frac{g^2}{4\pi} \quad (2.26)$$

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} - \beta_2 \frac{g^7}{(16\pi^2)^3} \quad (2.27)$$

and inserting these expansions into (2.24) gives:

$$U(\mu, M_W) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J_1 + \left(\frac{\alpha_s(\mu)}{4\pi} \right)^2 J_2 \right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right]^P \left[1 - \frac{\alpha_s(M_W)}{4\pi} J_1 - \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 (J_2 - J_1^2) \right] \quad (2.28)$$

where

$$P = \frac{\gamma^{(0)}}{2\beta_0}, \quad J_1 = \frac{P}{\beta_0} \beta_1 - \frac{\gamma^{(1)}}{2\beta_0}, \quad (2.29)$$

$$J_2 = \frac{P}{2\beta_0} \beta_2 + \frac{1}{2} \left(J_1^2 - \frac{\beta_1}{\beta_0} J_1 \right) - \frac{\gamma^{(2)}}{4\beta_0}. \quad (2.30)$$

General formulae for the evolution matrix $\hat{U}(\mu, M_W)$ in the case of operator mixing and valid also for electroweak effects at the NLO level can be found in [15]. The corresponding NNLO formulae are rather complicated and given in [31]. The leading logarithmic approximation corresponds to setting $J_1 = J_2 = 0$ in (2.28). In the NLO only $J_2 = 0$ and the last term in (2.28) has to be removed.

The coefficients β_i are given as follows

$$\beta_0 = \frac{33 - 2f}{3} \quad \beta_1 = \frac{306 - 38f}{3}, \quad (2.31)$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18}f + \frac{325}{54}f^2 \quad (2.32)$$

where f is the number of quark flavours.

The expansion for $C(M_W)$ is given by

$$C(M_W) = C_0 + \frac{\alpha_s(M_W)}{4\pi}C_1 + \left(\frac{\alpha_s(M_W)}{4\pi}\right)^2 C_2 \quad (2.33)$$

where C_0 , C_1 and C_2 depend generally on m_t , M_W , the masses of the new particles and the new parameters in the extensions of the SM. It should be stressed that the renormalization scheme dependence of C_1 and C_2 is canceled by the one of J_1 and J_2 in the last square bracket in (2.28) although at the NNLO level this cancellation is rather involved. The scheme dependence of J_1 and J_2 in the first square bracket in (2.28) is canceled by the scheme dependence of $\langle Q(\mu) \rangle$. The power P is scheme independent. The methods for the calculation of $\hat{U}(\mu, M_W)$ and the discussion of the cancellation of the μ - and renormalization scheme dependences are presented in detail in [16] and in the original papers where such calculations have been done.

When talking about the μ -dependence one should distinguish two types of dependences. First we have the μ dependence related to the renormalization of operators and present in the evolution matrix. This dependence arises in the presence of non-vanishing anomalous dimensions of the operators responsible for weak decays. But in the coefficients $C_i(M_W)$ in (2.33) also heavy quark masses are present that are running masses with their scale dependence governed by the anomalous dimension of the mass operator

$$\gamma_m(\alpha_s) = \gamma_m^{(0)} \frac{\alpha_s}{4\pi} + \gamma_m^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 + \gamma_m^{(2)} \left(\frac{\alpha_s}{4\pi}\right)^3 \quad (2.34)$$

with the coefficients $\gamma_m^{(i)}$ given as follows [32, 33, 34]

$$\gamma_m^{(0)} = 8, \quad \gamma_m^{(1)} = \frac{404 - 40f}{3} \quad (2.35)$$

$$\gamma_m^{(2)} = 2 \left[1249 - \left(\frac{2216}{27} + \frac{160}{3} \zeta(3) \right) f - \frac{140}{81} f^2 \right] \quad (2.36)$$

where $\zeta(3) \approx 1.202057$. These results are valid in the $\overline{\text{MS}}$ scheme. The four-loop contribution has been found in [35] but it is too complicated to be presented here. The most recent values of running quark and lepton masses can be found in [36].

If only the leading term C_0 is present the choice of the μ for the masses matters. This unphysical scale dependence is cancelled by the non-leading terms in (2.33). A detailed discussion of this issue can be found in [16] and we will return to it briefly below.

For later purposes it will be useful to generalize the formula (2.26) to include mixing between operators and $\mathcal{O}(\alpha)$ effects, where α is the QED coupling constant. This formula is relevant whenever also electroweak effects are present and electroweak penguin operators contribute. At the NNLO level in QCD but to leading order in α one has:

$$\hat{\gamma}(\alpha_s, \alpha) = \hat{\gamma}_s^{(0)} \frac{\alpha_s}{4\pi} + \hat{\gamma}_e^{(0)} \frac{\alpha}{4\pi} + \hat{\gamma}_s^{(1)} \left(\frac{\alpha_s}{4\pi} \right)^2 + \hat{\gamma}_{se}^{(1)} \frac{\alpha_s}{4\pi} \frac{\alpha}{4\pi} + \hat{\gamma}_s^{(2)} \left(\frac{\alpha_s}{4\pi} \right)^3 + \hat{\gamma}_{se}^{(2)} \left(\frac{\alpha_s}{4\pi} \right)^2 \frac{\alpha}{4\pi} \quad (2.37)$$

with $\hat{\gamma}_s^{(0)}$, $\hat{\gamma}_s^{(1)}$ and $\hat{\gamma}_s^{(2)}$ being anomalous dimension matrices that are generalizations of the corresponding coefficients in (2.26) to include mixing among operators under QCD renormalization. If $\mathcal{O}(\alpha)$ effects are included in the coefficients at scales $\mathcal{O}(M_W)$, the anomalous dimension matrix must also include $\mathcal{O}(\alpha)$ contributions which are represented by $\hat{\gamma}_e^{(0)}$, $\hat{\gamma}_{se}^{(1)}$ and $\hat{\gamma}_{se}^{(2)}$ at LO, NLO and NNLO, respectively.

The corresponding generalization of the Wilson coefficients in (2.33) takes the form

$$\vec{C}(M_W) = \vec{C}_0 + \frac{\alpha_s(M_W)}{4\pi} \vec{C}_1 + \left(\frac{\alpha_s(M_W)}{4\pi} \right)^2 \vec{C}_2 + \frac{\alpha}{4\pi} \vec{C}_1^{es} + \frac{\alpha_s(M_W)}{4\pi} \frac{\alpha}{4\pi} \vec{C}_2^{es} \quad (2.38)$$

where now the coefficients are column vectors and the evolution in (2.25) generalizes for $\alpha = 0$ to the one in (2.23). For $\alpha \neq 0$ similar formulae exist. We will give later the directions to papers, where explicit expressions for all these objects can be found.

2.4.2 Renormalization Scheme Dependence

As already stated above, beyond LO various quantities like Wilson coefficients and the anomalous dimensions depend on the renormalization scheme for operators. This dependence arises because the renormalization prescription involves an arbitrariness in the finite parts to be subtracted along with the ultraviolet singularities. Two different schemes are then related by a finite renormalization.

I have discussed this issue in detail in [16], in particular in Section 6.7 of these lectures. Here I just want to recall one NLO formula to which I will refer from time to time. It is a relation between anomalous dimension matrices in two different renormalization schemes:

$$\hat{\gamma}^{(0)'} = \hat{\gamma}^{(0)} \quad \hat{\gamma}^{(1)'} = \hat{\gamma}^{(1)} + [\Delta\hat{r}, \hat{\gamma}^{(0)}] + 2\beta_0 \Delta\hat{r}, \quad (2.39)$$

where the *prime* distinguishes the two schemes and $\Delta\hat{r}$ is a shift at $\mathcal{O}(\alpha_s)$ in the matrix elements of operators calculated in these two renormalization schemes:

$$\langle \vec{Q} \rangle' = (1 + \frac{\alpha_s}{4\pi} \Delta\hat{r}) \langle \vec{Q} \rangle, \quad \vec{C}' = (1 - \frac{\alpha_s}{4\pi} (\Delta\hat{r})^T) \vec{C}. \quad (2.40)$$

2.5 Inclusive Decays

So far I have discussed only *exclusive* decays. It turns out that in the case of *inclusive* decays of heavy mesons, like B -mesons, things turn out to be easier. In an inclusive decay one sums over all (or over a special class) of accessible final states and eventually one can show that the resulting branching ratio can be calculated in the expansion in inverse powers of m_b with the leading term described by the spectator model in which the B -meson decay is modelled by the decay of the b -quark. Very schematically one has then for the decay rate

$$\Gamma(B \rightarrow X) = \Gamma(b \rightarrow q) + \mathcal{O}\left(\frac{1}{m_b^2}\right). \quad (2.41)$$

This formula is known under the name of the Heavy Quark Expansion (HQE) [37]. Pedagogical reviews on this topic and heavy quark effective theories can be found in [37, 38, 39, 40, 41].

Since the leading term in this expansion represents the decay of the quark, it can be calculated in perturbation theory or more correctly in the renormalization group improved perturbation theory. It should be realized that also here the basic starting point is the effective Hamiltonian (2.1) and that the knowledge of the couplings $C_i(\mu)$ is essential for the evaluation of the leading term in (2.41). But there is an important difference relative to the exclusive case: the matrix elements of the operators Q_i can be “effectively” evaluated in perturbation theory. This means, in particular, that their μ and renormalization scheme dependences can be evaluated and the cancellation of these dependences by those present in $C_i(\mu)$ can be investigated.

Clearly in order to complete the evaluation of $\Gamma(B \rightarrow X)$ also the remaining terms in (2.41) have to be considered. These terms are of a non-perturbative origin, but fortunately they are suppressed often by two powers of m_b . They have been studied by several authors in the literature with the result that they affect various branching ratios by less than 10% and often by only a few percent. Consequently the inclusive decays give generally more precise theoretical predictions at present than the exclusive decays. On the other hand their measurements are harder. There are of course some important theoretical issues related to the validity of HQE in (2.41) which appear in the literature under the name of quark-hadron duality but I will not discuss them here.

The very rough appearance of the second term on the r.h.s of (2.41) totally underrepresents the efforts which have been made over many years to calculate these contributions. But as I was not involved in these efforts and they contain some non-perturbative aspects, I will not discuss them here. On the other hand we will summarize the status of the first term in (2.41) in some detail in Sections 6 and 8.

2.6 The Structure and the Status of the NLO and NNLO Corrections

2.6.1 General Comments

As we will see in the following sections during the last two decades the NLO corrections to $C_i(\mu)$ have been calculated within the SM for the most important and interesting decays. Also several NNLO calculations have been performed. In tables 2-6 we give references to all NLO and NNLO calculations within the SM done until the end of February 2011 that deal with the processes discussed by us. While these calculations improved considerably the precision of theoretical predictions in weak decays and can be considered as an important progress in this field, the pioneering LO calculations for current-current operators [42, 43], penguin operators [44, 45], $\Delta S = 2$ operators [46] and rare K decays [47] should not be forgotten.

2.6.2 Different Classes of QCD Corrections

The structure of QCD corrections to various decays depends on the decay considered. In particular the expansion in α_s can vary from decay to decay. Moreover even within a given decay the structure of QCD corrections to internal charm and top contributions differ from each other. Let us then classify various cases beginning with the simplest ones and systematically increasing the complexity.

Class 1

The simplest situation arises when there is only one contributing operator with a vanishing anomalous dimension and in addition there is no mixing of this operator with operators which have non-vanishing anomalous dimensions. Moreover the diagrams in the full theory from which this operator was born have only heavy internal particles like W^\pm , the top quark and generally heavy particle exchanges in the extensions of the SM.

This is the case of the operators $Q_{\nu\bar{\nu}}(B)$, $Q_{\mu\bar{\mu}}(B)$ in (2.13) and $\tilde{Q}_{\nu\bar{\nu}}(B)$, $\tilde{Q}_{\mu\bar{\mu}}(B)$ in (2.20) contributing to rare decays $B \rightarrow X_s \nu\bar{\nu}$, $B \rightarrow K^* \nu\bar{\nu}$, $B \rightarrow K \nu\bar{\nu}$ and $B_s \rightarrow \mu^+ \mu^-$. If the operators $Q_{\nu\bar{\nu}}(K)$ and $Q_{\mu\bar{\mu}}(K)$ in (2.14) and $\tilde{Q}_{\nu\bar{\nu}}(K)$ and $\tilde{Q}_{\mu\bar{\mu}}(K)$ in (2.20) originate in the internal top quark contributions and other heavy particle contributions to rare decays $K \rightarrow \pi \nu\bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ then also these contributions belong to this class. The case of internal charm quark contributions is classified separately below.

Denoting a given loop function in the absence of QCD corrections by $F_1(x)$, the decay amplitudes in this case have the following perturbative expansion in α_s

$$A_1 = F_1(x) + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2) \quad (2.42)$$

with α_s evaluated at the high scale where the operator has been generated. We drop the Lorentz structure for simplicity. Therefore $\mathcal{O}(\alpha_s)$ corrections are generally small and no large logarithm related to operator renormalization is present in them due to the vanishing of the anomalous dimension of the contributing operator. However $x = m_t^2/M_W^2$ depends on a scale μ_t present

in $m_t(\mu_t)$ with similar comments applying to other coloured heavy particles present beyond the SM. The corresponding logarithm involving this scale and present in the $\mathcal{O}(\alpha_s)$ correction in (2.42) cancels this scale dependence present in the leading term $F_1(x)$ so that up to higher order corrections A_1 is independent of μ_t . On the other hand the size of the $\mathcal{O}(\alpha_s)$ correction in (2.42) clearly depends on the chosen μ_t . It turns out that it is useful to set $\mu_t = m_t(m_t)$. Then the result can be summarized by

$$A_1 = F_1(x)\eta_{\text{QCD}} \quad (2.43)$$

with η_{QCD} close to unity and practically independent of the measured top quark mass. For other choices of μ_t the factor η_{QCD} can differ significantly from unity but then also the numerical value of $F_1(x)$ is different so that A_1 remains the same up to higher order corrections. This removal of order 10% dependence on μ_t in the LO formulae for rare K and B decays was the basic motivation for the calculations in [48, 49]. The QCD calculations in this class are described in Section 5 and the relevant references are collected in Table 4.

Class 2

This class is constructed from Class 1 by giving the operator an anomalous dimension but still requiring that it does not mix with other operators and all particles in loops generating this operator are heavy. This is the case of $\Delta S = 2$ and $\Delta B = 2$ operators in the SM when only top quark contributions in the box diagrams are considered. In this case we have

$$A_2 = F_2(x) * 1_{\text{QCD}} + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2), \quad (2.44)$$

where the funny 1_{QCD} represents the leading RG factor like the one involving P in (2.28). Now the $\mathcal{O}(\alpha_s)$ and higher order terms involve not only M_W but also low energy scale μ at the end of the RG evolution as seen explicitly in (2.28). Moreover the $\mathcal{O}(\alpha_s(M_W))$ correction involves now two logarithms multiplied by two different anomalous dimensions, one anomalous dimension of the mass operator related to the μ_t dependence present already in Class 1 and the second present in P involving the anomalous dimension of $Q(\Delta F = 2)$ in (2.11). The latter logarithm cancels the μ_W dependence present in the funny factor 1_{QCD} in (2.44) so that A_2 does not depend on the precise value of the scale at which the Wilson coefficients are defined. Again one can summarize the result schematically by

$$A_2 = F_2(x)\eta_{\text{QCD}}. \quad (2.45)$$

However, this time η_{QCD} can depart significantly from unity as summation of large logarithms in the process of RG evolution is involved. η_{QCD} depends as seen in (2.28) on the lower scale μ and this dependence cancels the one present in the hadronic matrix elements. This later dependence in η_{QCD} is often factored out so that the known factors $\eta_2 \equiv \eta_{tt}$ in ε_K and η_B in $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixing are μ -independent and this also applies to the \hat{B}_i factors that up to factors involving

weak decays constants represent hadronic matrix elements. Explicit expressions are given in Section 4.

This discussion applies also to the $\Delta F = 2$ operators in (2.15) except that now mixing under renormalization between operators Q_1^{LR} and Q_2^{LR} and similarly between Q_1^{SLL} and Q_2^{SLL} takes place. Explicit formulae for this case can be found in [50].

In some extensions of the SM FCNC operators are generated already at tree level but also in this case analogous discussion can be made.

The QCD calculations in this class are described in Section 4 and the relevant references are collected in Table 3.

Class 3

We next consider QCD corrections to charm contributions to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ decays. The corresponding operators are those of Class 1 but now a light charm quark mass is present in the loop. Consequently even without QCD corrections a large logarithm $\ln m_c/M_W$ is present and on the way to low scales bilocal operators enter the game. They undergo a rather complicated renormalization [51]. Only when the charm is integrated out we obtain the local operators $Q_{\nu\bar{\nu}}(K)$ and $Q_{\mu\bar{\mu}}(K)$ in (2.14). From the point of view of the renormalization group analysis, the expansion in α_s takes in this case the following form

$$A_3 = \mathcal{O}\left(\frac{1}{\alpha_s}\right) + \mathcal{O}(1) + \mathcal{O}(\alpha_s). \quad (2.46)$$

Thus the NLO corrections to the charm part of the amplitudes for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ amount to the $\mathcal{O}(1)$ term [51], while the NNLO corrections to the $\mathcal{O}(\alpha_s)$. Still to get this term three-loop diagrams have to be evaluated [31, 52]. The LO term has been calculated in [47].

The QCD calculations in this class are described in Section 5 and the relevant references are collected in Table 4.

Class 4

We next consider the set of $\Delta F = 1$ current-current operators in (2.5) that both have nonvanishing anomalous dimensions and mix under renormalization. After diagonalization of this system one gets two operators Q_{\pm} in (3.8) which evolve without mixing. The situation for each operator is then similar to Class 2 except that no box diagrams involving heavy particles have to be evaluated. Consequently the expansion is as follows

$$A_4 = 1_{\text{QCD}} + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2), \quad (2.47)$$

where α_s terms are evaluated both at M_W and the low scale μ according to the evolution in (2.28). The leading term is again the one involving P in (2.28).

The QCD calculations in this class are described in Section 3 and the relevant references are collected in Table 2.

Class 5

We next consider QCD penguin and electroweak penguin operators in (2.6)-(2.9) contributing to non-leptonic decays. These operators mix under QCD and QED renormalization and evidently the QCD penguin and electroweak penguin diagrams in the full theory are $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha)$, respectively. The Wilson coefficients of the corresponding operators after the top quark and W^\pm have been integrated out are also of the same order respectively. This mismatch of powers in α_s can be overcome in the process of renormalization group analysis by multiplying the electroweak operators by $1/\alpha_s$ and compensating this rescaling by multiplying their Wilson coefficients by α_s . As α is from the point of view of QCD a fixed number, the RG can now be performed as in class 2 except for the following changes. Q_1 and Q_2 operators have to be included as they mix into $Q_3 - Q_{10}$ operators affecting their QCD evolution. Thus we deal with 10×10 anomalous dimension matrices but because of the presence of electroweak penguins also terms $\mathcal{O}(\alpha\alpha_s)$ at NLO order have to be considered and $\mathcal{O}(\alpha\alpha_s^2)$ at NNLO in addition to the usual $\mathcal{O}(\alpha_s^2)$ and (α_s^3) terms, respectively. See (2.37).

We observe that now the formulae are a bit more involved but what is more important are the following facts which apply for instance to the evaluation of the ratio ε'/ε in $K_L \rightarrow \pi\pi$:

- At LO there is no top quark mass dependence nor any heavy particle mass dependence from penguin diagrams.
- At NLO these mass dependences enter for the first time.
- The NLO matching conditions for electroweak penguin operators do not involve QCD corrections to box and penguin diagrams and consequently the renormalization scale dependence in the top quark mass in these processes is not negligible at the NLO. In order to reduce this unphysical dependence, QCD corrections to the relevant box and penguin diagrams have to be computed. In the renormalization group improved perturbation theory these corrections are a part of NNLO corrections. In [53] such corrections have been computed for ε'/ε . Their inclusion allowed to reduce renormalization scheme dependence present in the electroweak penguin sector.

The QCD calculations in this class are described in Section 3 and the relevant references are collected in Table 2.

Class 6

We next come to magnetic penguin operators in (2.10) restricting the discussion to the $B \rightarrow X_s \gamma$ decay. These two operators mix under renormalization with each other and are also influenced by the mixing with the current-current operators and QCD penguin operators. Thus we deal here with a 8×8 anomalous dimension matrix.

I will report on the heroic efforts to calculate the QCD corrections to the $B \rightarrow X_s \gamma$ rate in Section 6. Let me here only write down symbolically the general structure of the amplitudes in

this class:

$$A_6 = F_6(x) * 1_{\text{QCD}} + \tilde{1}_{\text{QCD}} + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2), \quad (2.48)$$

with the first two terms representing LO and the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ terms NLO and NNLO corrections, respectively. The $\tilde{1}_{\text{QCD}}$ term represents the mixing of magnetic penguin operators with the current-current and QCD-penguin operators. In fact this term is responsible for a strong enhancement of the $B \rightarrow X_s \gamma$ decay rate.

The QCD calculations of $B \rightarrow X_s \gamma$ decay rate are described in Section 6 and the relevant references are collected in Table 5.

Class 7

In rare decays $K_L \rightarrow \pi^0 l^+ l^-$ and $B \rightarrow X_s l^+ l^-$ semileptonic operators in (2.12) are involved. In addition to these operators also current-current non-leptonic operators (2.5) and QCD penguin operators in (2.6) and (2.7) have to be taken into account. The electroweak penguins and magnetic penguins turn out to be irrelevant in $K_L \rightarrow \pi^0 l^+ l^-$ but the magnetic penguins have to be included in $B \rightarrow X_s l^+ l^-$.

The new feature is that although the operator Q_{9V} has no anomalous dimension by itself it mixes with the operators $Q_1 - Q_6$ and consequently a 7×7 anomalous dimension matrix has to be considered. The resulting structure of the coefficient C_{9V} is then

$$C_{9V} = \mathcal{O}\left(\frac{1}{\alpha_s}\right) + F(x) + \tilde{1}_{\text{QCD}} + \mathcal{O}(\alpha_s). \quad (2.49)$$

Consequently the structure is similar to Class 3 but at the NLO level in this case heavy quark mass dependence enters.

The operator Q_{10A} has no anomalous dimension and similarly to operators in (2.13) and (2.14) does not mix with anybody. Therefore its Wilson coefficient has the structure in (2.42). Evidently this operator enters the amplitude for $K_L \rightarrow \pi^0 l^+ l^-$ at the NLO level.

In the case of $B \rightarrow X_s l^+ l^-$ the situation is complicated by the presence of magnetic operators but the structure of the corresponding Wilson coefficients C_{9V} and C_{10A} is the same. The $\mathcal{O}(\alpha_s)$ corrections to the penguin and box diagram relevant for the NNLO evaluation of $B \rightarrow X_s l^+ l^-$ rate have been calculated in [55].

The QCD calculations of $K_L \rightarrow \pi^0 l^+ l^-$ and $B \rightarrow X_s l^+ l^-$ are described in Sections 7 and 8 respectively and the relevant references are collected in Table 6.

Class 8

Finally I give the structure of QCD corrections for the charm-charm ($\eta_1 \equiv \eta_{cc}$) and charm-top ($\eta_3 \equiv \eta_{ct}$) contributions to $\Delta S = 2$ Hamiltonian. The structure of these corrections differs from $\eta_2 \equiv \eta_{tt}$ discussed in Class 2 and also from each other but I think it is instructive to put them together in order to see the difference. We have

$$\eta_1 = (\alpha_s)^{P^+} (1_{\text{QCD}} + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_s^2)) \quad (2.50)$$

$$\eta_3 = (\alpha_s)^{P_+} \left(\frac{1}{\alpha_s} 1_{\text{QCD}} + \tilde{1}_{\text{QCD}} + \mathcal{O}(\alpha_s) \right) \quad (2.51)$$

The references to QCD calculations of η_1 and η_3 are collected in Table 3 where also several remarks on these calculations will be made.

2.6.3 Two-Loop Anomalous Dimensions Beyond the SM

In the extensions of the SM new operators are present. The two loop anomalous dimensions for the $\Delta F = 2$ and $\Delta F = 1$ four-quark dimension-six operators listed in (2.15) and (2.16) have been computed in [30, 29]. In [29] also the remaining anomalous dimensions of $\Delta F = 1$ four-quark operators can be found.

2.6.4 Two-Loop Electroweak Corrections

In order to reduce scheme and scale dependences related to the definition of electroweak parameters like $\sin^2 \theta_W$, and α_{QED} , two-loop electroweak contributions to rare decays have to be computed. For $K_L^0 \rightarrow \pi^0 \nu \bar{\nu}$, $B_{d,s} \rightarrow l^+ l^-$ and $B \rightarrow X_s \nu \bar{\nu}$ they can be found in [56, 57], for $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixing in [58] and for $B \rightarrow X_s \gamma$ in [59, 60, 61, 62, 63].

2.6.5 NLO QCD Calculations Beyond the SM

There exist also a number of partial or complete NLO QCD calculations within the Two-Higgs-Doublet Model (2HDM) and the MSSM. In the case of the Two-Higgs-Doublet Model such calculations for $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixing, $B \rightarrow X_s \gamma$ and $B \rightarrow X_s l^+ l^-$ can be found in [64], [55, 65, 66, 67, 68] and [69], respectively. In 2HDM also NNLO QCD Corrections to $B \rightarrow X_s \gamma$ have been recently calculated [70].

The corresponding NLO calculations for $B_{d,s}^0 - \bar{B}_{d,s}^0$ and $B \rightarrow X_s \gamma$ in the MSSM can be found in [71, 72, 73, 74] and [55, 75, 76, 77, 78, 79], respectively. The paper [55] gives also the results for $B \rightarrow X_s \text{gluon}$. In fact Bobeth, Misiak and Urban [55] present rather general formulae for Wilson coefficients relevant for $B \rightarrow X_s \gamma$ and $B \rightarrow X_s \text{gluon}$ evaluated at high scale (matching conditions) at the LO and NLO level that can be used for other extensions of the SM.

Finally, I would like to mention two calculations in which I took part: NLO QCD corrections to rare K and B decays in the MSSM at low $\tan \beta$ [80] and NNLO QCD corrections to $B \rightarrow X_s l^+ l^-$ in the MSSM [81].

2.6.6 Penguin-Box Expansion

The rare and CP violating decays of K and B mesons as well as ε_K , ε'/ε and $B_q^0 - \bar{B}_q^0$ mixings are governed in the SM by various penguin and box diagrams with internal top quark and charm quark exchanges. Some examples are shown in Fig. 1. Evaluating these diagrams one

finds a set of basic universal (process independent) m_t -dependent functions $F_r(x_t)$ [82, 16] where $x_t = m_t^2/M_W^2$. Explicit expressions for these functions can be found in [16].

It is useful to express the OPE formula (2.2) directly in terms of the functions $F_r(x_t)$ [83]:

$$A(M \rightarrow F) = P_0(M \rightarrow F) + \sum_r P_r(M \rightarrow F) F_r(x_t), \quad (2.52)$$

where the sum runs over all possible functions contributing to a given amplitude. P_0 summarizes contributions stemming from internal charm quark. In the OPE formula (2.2), the functions $F_r(x_t)$ are hidden in the initial conditions for $C_i(\mu)$ represented by $\vec{C}(M_W)$ in (2.23).

The coefficients P_0 and P_r are process dependent and include QCD corrections contained in the evolution matrix $\hat{U}(\mu, M_W)$. They depend also on hadronic matrix elements of local operators and the relevant CKM factors. An efficient and straightforward method for finding the coefficients P_r is presented in [83]. As the expansion in (2.52) involves basic one-loop functions from penguin and box diagrams it was naturally given the name of the *Penguin-Box Expansion* (PBE).

Generally, several basic functions contribute to a given decay, although decays exist which depend only on a single function. We have the following correspondence between the most interesting FCNC processes and the basic functions within models with constrained Minimal Flavour Violation:

$K^0 - \bar{K}^0$ -mixing (ε_K)	$S(v)$
$B_{d,s}^0 - \bar{B}_{d,s}^0$ -mixing ($\Delta M_{s,d}$)	$S(v)$
$K \rightarrow \pi \nu \bar{\nu}$, $B \rightarrow X_{d,s} \nu \bar{\nu}$	$X(v)$
$K_L \rightarrow \mu \bar{\mu}$, $B_{d,s} \rightarrow l \bar{l}$	$Y(v)$
$K_L \rightarrow \pi^0 e^+ e^-$	$Y(v), Z(v), E(v)$
ε' , Nonleptonic $\Delta B = 1$, $\Delta S = 1$	$X(v), Y(v), Z(v), E(v)$
$B \rightarrow X_s \gamma$	$D'(v), E'(v)$
$B \rightarrow X_s$ gluon	$E'(v)$
$B \rightarrow X_s l^+ l^-$	$Y(v), Z(v), E(v), D'(v), E'(v),$

where v denotes collectively the arguments of a given function with $v = x_t$ in the SM. In these models the operator structure of the SM remains intact and NP modifies only the basic functions. It should be mentioned that this correspondence is strictly valid in LO. At NLO in processes in which mixing between different operators is present new loop functions can contribute to a given process but these contributions are generally small.

Originally PBE was designed to expose the m_t -dependence of FCNC processes [83] which was hidden in the Wilson coefficients. In particular in the case of ε'/ε , where many of these functions enter, this turned out to be very useful. After the top quark mass has been measured precisely this role of PBE is less important. On the other hand, PBE is very well suited for the study of the extensions of the SM in which new particles are exchanged in the loops. If

there are no new local operators beyond those present in the SM the mere change is to modify the functions $F_r(x_t)$ which now acquire the dependence on the masses of new particles such as charged Higgs particles and supersymmetric particles. The process dependent coefficients P_0 and P_r remain unchanged. The effects of new physics can then be seen transparently. Many examples of the applications of PBE can be found in the literature. In particular in the last decade we have used this method for the study of FCNC processes in the MSSM, a model with a universal extra dimension, littlest Higgs model (LH), littlest Higgs model with T parity (LHT), MSSM at low $\tan\beta$ and also the SM with four generations. In these papers compilations of the functions F_r in a given model can be found. A complete list of references can be found in [84].

One virtue of this method is a transparent study of the departure from MFV. In this framework, as discussed in detail in [85], the basic loop functions are universal with respect to the system considered. Indeed as seen above the same function $S(v)$ enters ε_K and $B_{d,s} - \bar{B}_{d,s}$ mixings. Similarly the same function $X(v)$ enters $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $B \rightarrow X_{d,s} \nu \bar{\nu}$ decays. Indeed in MFV the flavour dependence resides fully in the CKM matrix elements. Moreover, all these functions are real as the sole complex CP-violating phase resides in the CKM matrix and in flavour blind CP phases. In the presence of new sources of flavour and CP violation things are different:

- The universality in question is broken and many MFV relations between various branching ratios are generally violated.
- The basic functions become complex quantities leading often to new CP-violating effects not encountered in the CKM framework.

All these new effects can be transparently seen in this framework.

If new effective operators with different Dirac and colour structures are present, new functions multiplied by new coefficients $P_r(M \rightarrow F)$ contribute to (2.52). Still one is free to add these contributions to the functions of the SM, but then the dependence on the new $P_r(M \rightarrow F)$ is included in the basic functions. This is not always convenient if one wants to see explicitly the effects of new right-handed operators or scalar operators Q_i^{LR} , Q_i^{SLL} and Q_i^{SRR} given previously. Therefore in this case it is better to proceed as follows.

We can start with (2.2) but instead of evaluating it at the low energy scale we choose for μ the high energy scale to be called μ_H at which heavy particles are integrated out. Then absorbing $G_F/\sqrt{2}$ and V_{CKM}^i in the Wilson coefficients $C_i(\mu_H)$ the amplitude for $M - \bar{M}$ mixing ($M = K, B_d, B_s$) is simply given by

$$A(M \rightarrow \bar{M}) = \sum_{i,a} C_i^a(\mu_H) \langle \bar{M} | Q_i^a(\mu_H) | M \rangle. \quad (2.53)$$

Here the sum runs over all the operators in (2.15), that is $i = 1, 2$ and $a = VLL, VRR, LR, \dots$. The matrix elements for $B_d - \bar{B}_d$ mixing are for instance given as follows [50]

$$\langle \bar{B}_d^0 | Q_i^a(\mu_H) | B_d^0 \rangle = \frac{2}{3} M_{B_d}^2 F_{B_d}^2 P_i^a(B_d), \quad (2.54)$$

where the coefficients $P_i^a(B_d)$ collect compactly all RG effects from scales below μ_H as well as hadronic matrix elements obtained by lattice methods at low energy scales. Analytic formulae for these coefficients are given in [50] while the recent applications of this method can be found in [86, 87, 88]. As the Wilson coefficients $C_i(\mu_H)$ depend directly on the loop functions and fundamental parameters of a given theory, this dependence can be exhibited if necessary.

Again as in the case of PBE the virtue of using high energy scale rather than the low energy scale is that the coefficients $P_i^a(M)$ can be evaluated once for all if the hadronic matrix elements are known.

The following points should be emphasized:

- The expressions (2.53) and (2.54) are valid for any model with the model dependence entering only the Wilson coefficients $C_i^a(\mu_H)$, which generally also depend on the meson system considered. In particular, they are valid both within and beyond the MFV framework. In MFV models CKM factors and Yukawa couplings define the flavour dependence of these coefficients, while in non-MFV models additional flavour structures are present in $C_i^a(\mu_H)$.
- The coefficients P_i^a are model independent and include the renormalization group evolution from high scale μ_H down to low energy $\mathcal{O}(\mu_K, \mu_B)$. As the physics cannot depend on the renormalization scale μ_H , the P_i^a depend also on μ_H so that the scale dependence present in P_i^a is canceled by the one in C_i^a . Explicit formulae for μ_H dependence of P_i^a can be found in [50]. It should be stressed that here we are talking about logarithmic dependence on μ_H . The power-like dependence (such as $1/M_H^2, \dots$) is present only in the C_i^a .
- The P_i^a depend however on the system considered as the hadronic matrix elements of the operators in (2.53) relevant for instance for $K^0 - \bar{K}^0$ mixing differ from the matrix elements of analogous operators relevant for $B_{s,d}^0 - \bar{B}_{s,d}^0$ systems. Moreover whereas the RG evolution in the latter systems stops at $\mu_B = \mathcal{O}(M_B)$, in the case of $K^0 - \bar{K}^0$ system it is continued down to $\mu_K \sim 2$ GeV, where the hadronic matrix elements are evaluated by lattice methods.

After this rather heavy material we are ready to return to the main story of this paper.

3 $\Delta S = 1$ and $\Delta B = 1$ Non-Leptonic and Semi-Leptonic Decays

3.1 Effective Hamiltonians

The effective Hamiltonian for non-leptonic $\Delta S = 1$ transitions is given in the SM as follows:

$$\mathcal{H}_{\text{eff}}(\Delta S = 1) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} (z_i(\mu) + \tau y_i(\mu)) Q_i(\mu) \quad (3.1)$$

with $\tau = -V_{ts}^* V_{td} / (V_{us}^* V_{ud})$. The operators Q_i are the analogues of the ones given in (2.5)-(2.9). In the case of $\Delta B = 1$ transitions the flavours have to be changed appropriately and the effective hamiltonian is usually written as follows

$$H_{eff}(\Delta B = 1) = \frac{G_F}{\sqrt{2}} \left[\lambda_u (C_1(\mu_b) Q_1^u + C_2(\mu_b) Q_2^u) + \lambda_c (C_1(\mu_b) Q_1^c + C_2(\mu_b) Q_2^c) - \lambda_t \sum_{i=3}^{10,8G} C_i(\mu) Q_i \right], \quad (3.2)$$

where

$$\lambda_q = V_{qs}^* V_{qb} \quad (3.3)$$

and

$$Q_1^q = (\bar{q}_\alpha b_\beta)_{V-A} (\bar{s}_\beta q_\alpha)_{V-A}, \quad Q_2^q = (\bar{q}_\alpha b_\alpha)_{V-A} (\bar{s}_\beta q_\beta)_{V-A}. \quad (3.4)$$

In particular $Q_i^c = Q_i$ in (2.5).

3.2 Current-Current Operators

Let me begin our NLO story with the first climb within the MNLC which Peter Weisz and myself started in December 1988: the calculation of two-loop anomalous dimensions of Q_1 and Q_2 operators. It involves 28 two-loop diagrams shown in Fig. 3 and the corresponding counter terms. We decided to perform it in three schemes for γ_5 : anticommuting (NDR), 't Hoof-Veltman scheme and the DRED scheme used by the Italian group in 1981. I am sure that presently a calculation of that type is done fully automatically by means of appropriate computer programs but in the winter 1988/1989 and in the spring of 1989 we did it almost entirely by hand. This has the advantage that each step of the calculation can be followed and enjoyed in contrast to looking constantly at the computer.

Working then entirely by hand the first step is the calculation of two-loop momentum integrals keeping $1/\varepsilon^2$ and $1/\varepsilon$ terms. In doing this, it is useful to factor out any Dirac and colour structures. In this manner the results for the two-loop integrals in question can be used for the calculations of two-loop anomalous dimensions of other operators and in different renormalization schemes in which γ_5 is treated differently. This first step is rather tedious but straightforward and it is not surprising that working independently we obtained the same results for all 28 diagrams already in the first comparison. I have used these results for the calculation of two-loop anomalous dimensions of other operators in 1991 and 1999. I will discuss these calculations later on. In calculating colour factors I found the paper [89] very useful.

The last step of the calculation, the manipulation of Dirac structures turned out to be the crucial part of our work. When calculating the subdiagrams in Fig. 2 we encountered structures

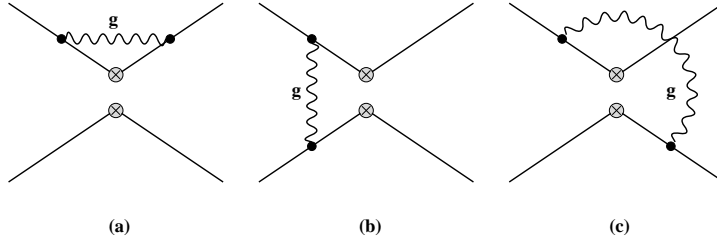


Figure 2: One loop current-current diagrams that contribute to one-loop anomalous dimensions and enter as subdiagrams the two-loop calculations of two-loop anomalous dimensions. The 4-vertex “ \otimes ” denotes the insertion of a 4-fermion operator Q_i .

like

$$\Gamma_\nu \gamma_\rho \gamma_\mu \otimes \Gamma^\nu \gamma^\rho \gamma^\mu, \quad \Gamma_\nu = \gamma_\nu (1 - \gamma_5) \quad (3.5)$$

that we had to reduce to the operators Q_1 and Q_2 which have the structure $\Gamma \otimes \Gamma$. At one-loop one can do this in $D = 4$ dimensions as $1/\epsilon$ is the leading singularity, but in a two-loop calculation the $1/\epsilon$ singularity, from which the anomalous dimensions are extracted, is next-to-leading. Consequently $\mathcal{O}(\epsilon)$ terms in Dirac structures multiplied by the leading $1/\epsilon^2$ singularity from the momentum integrals have an impact on two-loop anomalous dimensions and have to be taken properly into account.

As the first scheme we considered the one with anticommuting γ_5 in $D \neq 4$ dimensions giving it the name NDR (naive dimensional regularization). In order to reduce the structures like the one in (3.5) to Q_1 and Q_2 we first followed the procedure of Tracas and Vlachos [10] who simply wrote

$$\Gamma_\nu \gamma_\rho \gamma_\mu \otimes \Gamma^\nu \gamma^\rho \gamma^\mu = A \Gamma_\nu \otimes \Gamma^\nu \quad (3.6)$$

and found the coefficient A by replacing \otimes by γ_τ and contracting the Dirac indices on both sides. This procedure, to be called the “Greek method” in what follows, gives $A = 4(4 - \epsilon)$ and consequently

$$\Gamma_\nu \gamma_\rho \gamma_\mu \otimes \Gamma^\nu \gamma^\rho \gamma^\mu = 4(4 - \epsilon) \Gamma_\nu \otimes \Gamma^\nu. \quad (3.7)$$

This method is very efficient and can be applied to Dirac structures with many γ_μ that appear at two-loop level. It can also be easily generalized to other operators. For instance in the case of $\gamma_\mu (1 - \gamma_5) \otimes \gamma^\mu (1 + \gamma_5)$ one should replace \otimes by 1.

Applying this method to 28 diagrams in question and using it also for the counter diagrams we could readily find the total $1/\epsilon$ singularity. Subsequently including the two-loop wave function renormalization for the external quarks we found the two-loop anomalous dimension matrix in

the basis

$$Q_+ = \frac{Q_2 + Q_1}{2}, \quad Q_- = \frac{Q_2 - Q_1}{2} . \quad (3.8)$$

We expected this matrix to be diagonal but we both found it to contain non-diagonal terms. There was a hope for an hour. We disagreed on four diagrams. After a fortunate draw 2 : 2 we were certain that our calculation was algebraically correct but unfortunately the unwanted non-diagonal terms were still there. The only place, we could think, something went wrong was the “Greek method” for the reduction of the complicated Dirac structures to the physical operators Q_1 and Q_2 as given in (3.7).

Indeed in addition to the physical operators $Q_{1,2}$ on the r.h.s of (3.7) one could have other operators with different Dirac structures that vanish in $D = 4$. They had to vanish in $D = 4$ because in $D = 4$ the formula (3.7) is correct as can be easily checked by using standard manipulations of Dirac matrices. Peter called these operators “effervescent” operators. I did not object to this name. How could I? After all he is English, not me. Later we were told that the proper name is “evanescent” and we used this name in the following papers but in our first paper published in Nucl. Phys. B still “effervescent” operators appear [90]. Amusingly our Rome competitors used our wording still several years later [181].

Explicit expressions for the evanescent operators can be worked out. We have done it in our paper. But in practice it is more convenient to define them simply as the difference between the r.h.s. and l.h.s. of (3.7) and to insert them like that in the relevant two-loop diagrams. Therefore instead of (3.7) we have

$$\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu = 4(4 - \epsilon)\Gamma \otimes \Gamma + E^{\text{NDR}} \quad (3.9)$$

with the evanescent operator E^{NDR} defined simply by this equation. As discussed in detail in [91, 92] this is not the only possible definition of evanescent operators but possibly the most convenient one. The unphysical arbitrariness in the definition of evanescent operators has also been emphasized by Jamin and Pich [109].

Having identified the possible origin of our problems we have incorporated the evanescent operators into our calculation. In particular we derived, to my knowledge for the first time, formulae that allow the extraction of the two-loop anomalous dimensions of physical operators in the presence of evanescent operators. The outcome of these efforts is section 4 of our paper. We have invested plenty of time in writing this section but apparently several of my colleagues had a difficult time in following it. I tried to improve on it in my Les Houches lectures [16], where a systematic procedure for the inclusion of evanescent operators in the calculation of two loop anomalous dimensions of local operators can be found.

Having the full machinery for the evaluation of the contributions of evanescent operators at hand, we could now find that their presence not only modified the diagonal terms in the 2×2 matrix in question but also canceled the off-diagonal terms. We were now sure that the first

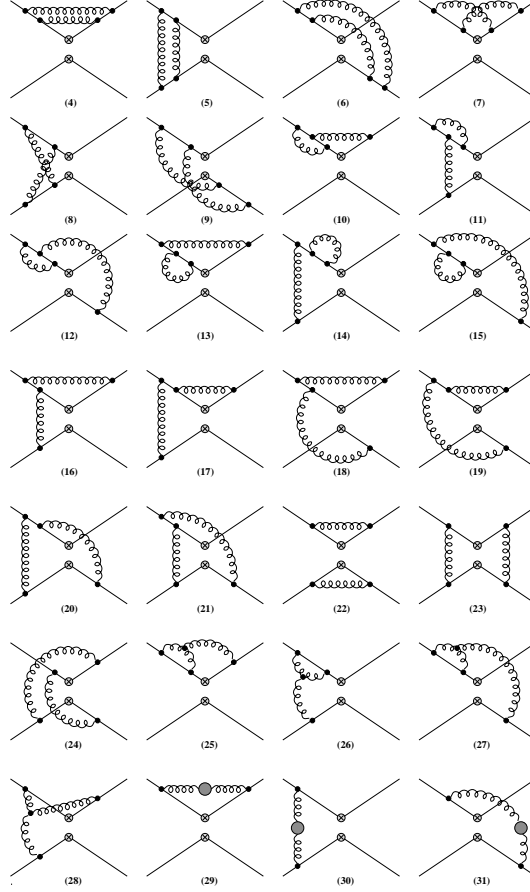


Figure 3: Two-loop current-current diagrams contributing to $\hat{\gamma}_s^{(1)}$. The curled lines denote gluons. The 4-vertices “ $\otimes \otimes$ ” denote standard operator insertions. In addition shaded blobs stand for self-energy insertions. Possible left-right or up-down reflected diagrams are not shown.

NLO summit has been conquered: we knew the (Q_+, Q_-) or equivalently the (Q_1, Q_2) matrix at the two-loop level in the NDR scheme. As at no place in the calculation it was necessary to evaluate the dangerous traces $Tr(\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\lambda \gamma_5)$, we were quite confident that this result was correct.

The calculation in the 't Hooft-Veltman (HV) scheme for γ_5 was technically more difficult because of the horrible Dirac algebra for which we had to use a computer program written by Peter. As I did the NDR calculation entirely by hand, including Dirac algebra, I could test the correctness of this program. Otherwise we did not encounter any obstacles and we soon had the two-loop anomalous dimension matrix of (Q_1, Q_2) in the HV scheme. By calculating the one-loop diagrams in Fig. 2 we found the matrix $\Delta\hat{r}$ in (2.40), relating the matrix elements of the operators in question in the NDR and HV schemes. Inserting it in (2.39) we could indeed verify

that our results for the two-loop anomalous dimensions in these two schemes were compatible with each other.

Strictly speaking this was the end of the story as we had the two-loop anomalous dimension matrix in the HV scheme that did not have any mathematical inconsistencies related to γ_5 . We have also demonstrated that at least in this case a consistent calculation in the simpler NDR scheme could be made. Still we were curious whether the calculation of the Italian pioneers in 1981 was compatible with our results. Actually it would suffice to calculate the relevant matrix $\Delta\hat{r}$ relating HV or NDR scheme to the DRED scheme to find out that the calculation of 1981 was correct, but for reasons that I do not understand today, we repeated the 1981 calculation confirming diagram by diagram the results of the Italian team. Most probably we were feeling very strong in such calculations and we were simply delighted in producing results for these 28 diagrams in a different renormalization scheme.

We have submitted our paper [90] to Nucl. Phys. B in June 1989 and sent our preprint to CERN and SLAC libraries. One should recall that in 1989 the Los Alamos archive did not exist yet. Few weeks later our preprint has been distributed by ordinary mail around the world.

At the end of July 1989 I attended the Photon-Lepton conference in Stanford. On the first day of the conference the two Guidos of the 1981 team congratulated me because of our paper. They were truly delighted. They were apparently not sure that their paper was correct. Indeed the calculation in the DRED scheme is very involved as also some aspects of QCD coupling renormalization have to be modified.

At the same conference I met for the first time Matthias Jamin, who just got his PhD in Heidelberg and was supposed to join my group in Munich two months later. I told him about the MNLC and he immediately became the third member of the club. Already in October of the same year we were climbing together the *second* NLO summit, the QCD corrections to $\Delta F = 2$ processes that I will describe briefly in the next section. For the time being I will continue with the $\Delta F = 1$ effective Hamiltonian for non-leptonic decays including now the QCD and electroweak penguin operators.

In Table 2 we collect the references to the papers which calculated NLO and NNLO corrections to $\Delta F = 1$ processes except for rare and radiative decays discussed in Sections 6–8. Two-Body B Decays in QCD Factorization (QCDF) approach are discussed in Section 9. I thank Gerhard Buchalla for helping me in collecting the references to NLO and NNLO calculations in QCDF given in this table.

3.3 QCD Penguin Operators

In the fall of 1990, after two successful expeditions, it was time to return to the QCD penguin operators that were the main topic of the seminal supper with Guido Martinelli in the Ringberg castle two and a half years before. There were no signs coming from Rome that the Italian team was making any progress on penguins but I started worrying that they were far ahead of

Table 2: NLO and NNLO Calculations for Non-leptonic and Semi-Leptonic $\Delta F = 1$ Transitions

Decay	NLO	NNLO
Current-Current (Q_1, Q_2)	[5, 90]	[93]
QCD penguins (Q_3, Q_4, Q_5, Q_6)	[94, 95, 96, 97, 98, 99],[100]	[93]
electroweak penguins (Q_7, Q_8, Q_9, Q_{10})	[101, 96, 97, 98]	[53]
$Br(B)_{SL}$	[102, 103, 104, 105]	
inclusive non-leptonic decays	[5, 6, 106, 107, 108]; [109]	
Two-Body B-Decays in QCDF	[279]–[289]	[295]–[303]
Current-Current (BSM)	[30, 29]	
Penguins (BSM)	[29]	
Semi-Leptonic B Decays ($ V_{cb} , V_{ub} $)	[121]–[126],[134, 135]	[126]–[133],[136]–[139]

us. For this reason I decided to increase our team. Markus Lautenbacher, my former diploma student and since April 1990 my PhD student, became the fourth member of the MNLC and its first PhD student. Markus did not have any experience with two-loop calculations but his high computer skills and an impressive discipline in doing research convinced me that he would be a great help in our project. Our first goal was the calculation of the 6×6 two-loop anomalous dimension matrix $\hat{\gamma}_s^{(1)}$ describing the mixing under renormalization of the operators Q_1, Q_2, \dots, Q_6 in the NDR and HV schemes. The calculation of $\hat{\gamma}_s^{(1)}$ involves the insertions of all these operators into vertex diagrams considered already by Peter and myself in our first paper and into two-loop penguin diagrams in Fig. 4 to be considered for the first time. The latter diagrams do not have any impact on the sector (Q_1, Q_2) so that the corresponding 2×2 submatrix of $\hat{\gamma}_s^{(1)}$ calculated by Peter and myself remained untouched.

In the first month I worked closely with Markus helping him in making first steps on this new ground, whereas Peter and Matthias worked independently by themselves. Later Matthias and Markus worked closely together and constructed an efficient program for Dirac algebra manipulations in $D \neq 4$ in the NDR and HV schemes [110]. This program written in Mathematica became an important part of our project in particular in the case of the HV scheme and even I used it in spite of my previous comments on computer manipulations. In this scheme the calculations of two-loop penguin diagrams by hand were prohibitive as even computer manipulations required in 1990 a good PC. The evaluation of two-loop momentum integrals in a few penguin diagrams that we did mostly by hand turned out to be rather involved and a method by Peter to find the coefficients of the divergences in these particular diagrams numerically was very helpful. The corresponding calculation of the vertex diagrams was simple as we had already all integrals from [90]. Only the Dirac structures were different.

There were two new features with respect to the calculation of the (Q_1, Q_2) system of 1989.

First we had to face the dangerous traces $Tr(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda\gamma_5)$. In the HV scheme they can be straightforwardly evaluated but their evaluation in the NDR scheme could lead to wrong results. For a few weeks we thought that we had to abandon the calculation in the NDR scheme but at the end we solved the problem in two ways. My solution was to work, dependently on the diagram considered, with a second operator basis $\{\tilde{Q}_i\}$ with \tilde{Q}_i being Fierz transformed operators of Q_i . With the help of these operators it was possible to avoid the appearance of the dangerous traces. However, simply replacing Q_i by \tilde{Q}_i in order to avoid dangerous traces with γ_5 and inserting it into a two-loop penguin diagram would eventually give the wrong result for $\hat{\gamma}_s^{(1)}$. This is the second new feature of the presence of penguin diagrams: the insertion of $(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{Q}_4)$ into a two-loop diagram gives different result from the insertion of (Q_1, Q_2, Q_3, Q_4) and consequently the two-loop mixing between (Q_1, Q_2) and (Q_3, Q_4) differs from the one between $(\tilde{Q}_1, \tilde{Q}_2)$ and $(\tilde{Q}_3, \tilde{Q}_4)$. Fortunately, similarly to the renormalization scheme dependence of $\hat{\gamma}_s^{(1)}$, the difference in question could be found by performing a one-loop calculation that did not involve dangerous traces with γ_5 . Incorporating this difference properly into the calculation that involved both the original operators and their Fierz transforms allowed then to obtain $\hat{\gamma}_s^{(1)}$ for the original basis in the NDR scheme without any problems with γ_5 .

This procedure is described in detail in [95]. In the case of the evaluation of the two-loop anomalous dimensions of electroweak penguin operators it had to be generalized to include more bases because of the more complicated flavour structure of these operators. This is described in [101]. This procedure was followed by Matthias and Markus. Peter succeeded to avoid the dangerous traces in a different manner but I do not remember how. This is however immaterial as our independent results for $\hat{\gamma}_s^{(1)}$ in the NDR scheme agreed with each other.

I must admit that at that time I was very satisfied with my procedure of working simultaneously in $D \neq 4$ with the original basis and the Fierz transformed basis and making finite renormalizations through one-loop calculations at the end as explained above. However, in 1994 a more elegant and a more systematic procedure with the same outcome has been proposed by Matthias Jamin and Toni Pich [109] in the course of their NLO analysis of inclusive $\Delta F = 1$ transitions. Basically one can make four dimensional Fierz transformations in a $D \neq 4$ calculation provided the evanescent operators that vanish in $D = 4$ under the Fierz transformation are also included in the analysis. They are simply given by the difference of a given operator and its Fierz transformed operator. This method has been rediscovered by Mikolaj Misiak and Jörg Urban in the process of another NLO climb that I did with them in 1999. I will return to it in Section 3.5.

Still my cooking recipe that involved calculating the differences between the one-loop insertion of an operator and of its Fierz transformed operator is very useful for finding out whether the evanescent operators of that type are relevant for the calculation of two-loop anomalous dimensions or not. If this difference vanishes, the evanescent operators in question do not contribute. This is the case of all operators Q_1, \dots, Q_{10} inserted into the current-current diagrams

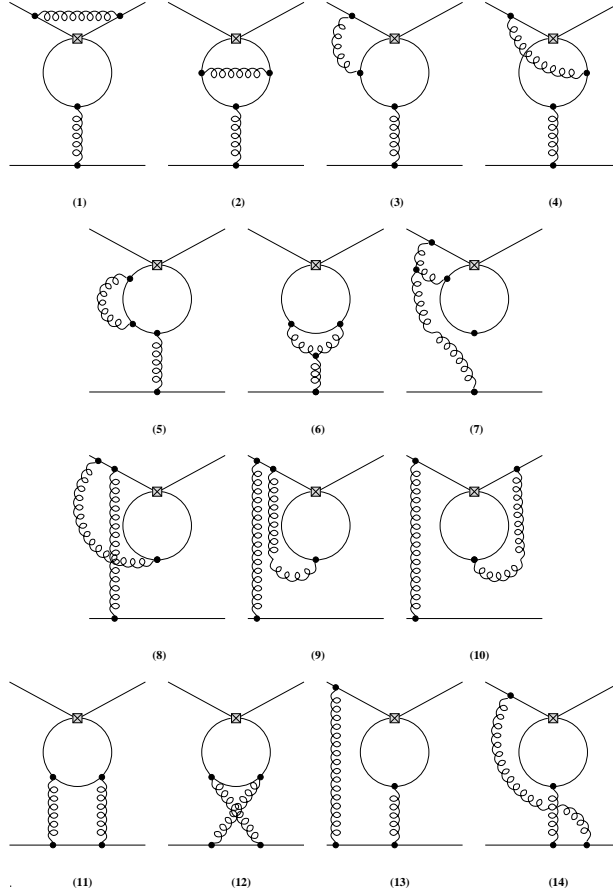


Figure 4: Two-loop penguin diagrams contributing to $\hat{\gamma}_s^{(1)}$. The curled lines denote gluons. Square-vertices stand for penguin insertions. Possible left-right reflected diagrams are not shown.

and of Q_5, \dots, Q_8 inserted into the penguin diagrams.

We thus had in the Spring of 1991 the full 6×6 anomalous dimension matrix $\hat{\gamma}_s$ at $\mathcal{O}(\alpha_s^2)$ in the NDR scheme.[¶] The calculation in the HV scheme was time consuming because of the difficult Dirac algebra but with the computer program developed by Markus and Matthias we could calculate $\hat{\gamma}_s^{(1)}$ in the HV scheme both directly by calculating the traces with γ_5 and by using my procedure discussed above, obtaining the same result. Finally calculating the relevant one loop shift $\Delta\hat{r}$ we verified that our results for $\hat{\gamma}_s^{(1)}$ in the NDR and HV schemes were consistent with each other.

Next we calculated the initial conditions for the Wilson coefficient functions at $\mu = \mathcal{O}(M_W)$ both in the NDR and the HV scheme and verified that the scheme dependence of these coefficients

[¶]It's generalization to 10×10 matrix is discussed in the next subsection.

cancelled the one of the evolution matrices at $\mu = \mathcal{O}(M_W)$ as explained in section 2.

Thus the third NLO summit has been reached. As usually reaching a summit a photo is taken, I thought it was appropriate to show the resulting two-loop anomalous dimension matrix $\hat{\gamma}_s^{(1)}$ in the NDR scheme:

$$\begin{pmatrix} -\frac{21}{2} - \frac{2f}{9} & \frac{7}{2} + \frac{2f}{3} & \frac{79}{9} & -\frac{7}{3} & -\frac{65}{9} & -\frac{7}{3} \\ \frac{7}{2} + \frac{2f}{3} & -\frac{21}{2} - \frac{2f}{9} & -\frac{202}{243} & \frac{1354}{81} & -\frac{1192}{243} & \frac{904}{81} \\ 0 & 0 & -\frac{5911}{486} + \frac{71f}{9} & \frac{5983}{162} + \frac{f}{3} & -\frac{2384}{243} - \frac{71f}{9} & \frac{1808}{81} - \frac{f}{3} \\ 0 & 0 & \frac{379}{18} + \frac{56f}{243} & -\frac{91}{6} + \frac{808f}{81} & -\frac{130}{9} - \frac{502f}{243} & -\frac{14}{3} + \frac{646f}{81} \\ 0 & 0 & \frac{-61f}{9} & \frac{-11f}{3} & \frac{71}{3} + \frac{61f}{9} & -99 + \frac{11f}{3} \\ 0 & 0 & \frac{-682f}{243} & \frac{106f}{81} & -\frac{225}{2} + \frac{1676f}{243} & -\frac{1343}{6} + \frac{1348f}{81} \end{pmatrix} \quad (3.10)$$

This matrix looks truly horrible and I will in what follows refrain from showing other photos of this type. In fact after the inclusion of electroweak penguins and going to three loops the results, although very impressive, cannot be easily digested. On the other hand the 2×2 submatrix in the upper left corner, our first summit, conquered by Peter Weisz and myself in June of 1989 looks beautiful and simple.

Our paper has been submitted to Nucl. Phys. B in May 1991 [94]. In addition to the 6×6 matrices and the Wilson coefficients of (Q_1, Q_2, \dots, Q_6) in the NDR and HV schemes for both $\Delta S = 1$ and $\Delta B = 1$ decays it contained general expressions for the evolution matrices $\hat{U}(\mu_1, \mu_2)$ including NLO corrections. In order to derive these expressions we have used the general all order formulae of my 1980 review on asymptotic freedom in deep inelastic scattering [111]. Finally we have demonstrated the scheme independence of the resulting decay amplitudes. Thus at last, three years after the Ringberg workshop, the Wilson coefficients of current-current and QCD penguin operators were known at NLO in the NDR and HV schemes.

I have presented these results in a parallel session at the joined Photon-Lepton and European Physical Society Meeting that in 1991 took place in Geneva, Switzerland [112]. Rather disappointingly only few of my colleagues appreciated these results. In particular Eduardo de Rafael thought it was an overkill in view of the uncertainties in the hadronic matrix elements of the operators in question. Eduardo got interested in our work only ten years later, when he wanted to know more about the role of evanescent operators and Fierz relations in our calculations that he and his collaborators wanted to combine with their calculations of hadronic matrix elements within the large N approach.

Also to my great surprise there was essentially no reaction from Guido Martinelli. He only informed me that his PhD students are working on this project as well and that in order to complete the project one needs the hadronic matrix elements of QCD penguin operators. In

fact in 1991 I knew these matrix elements in the context of $1/N$ expansion developed with Bardeen and Gérard several years earlier but Guido meant here the ones obtained by lattice methods. Unfortunately Guido's dream to achieve the latter result did not materialize until today. Hopefully in this decade we will know whether lattice results for the QCD-penguin hadronic parameter B_6 will be very close to large- N values as it turned out to be the case of \hat{B}_K relevant for the CP-violating parameter ε_K in $K_L \rightarrow \pi\pi$ [113].

As far as QCD penguins are concerned the first NLO analysis of penguin induced B -decays and the related CP-asymmetries using our two-loop results has been performed by my student, Robert Fleischer, in the Summer of 1992 [100]. Robert combined his one-loop calculations of matrix elements with the two-loop anomalous dimensions discussed above and demonstrated the scheme independence of the final result. It was his Diploma thesis. During his Phd studies Robert fell in love with electroweak penguins and investigated their role in non-leptonic decays. Therefore he did not have time to participate in the subsequent papers on NLO QCD corrections to weak decays except for his second paper in [100] which is a proof of his new interests in 1994.

Further progress in evaluation of the 6×6 anomalous dimension matrix for the operators Q_1, \dots, Q_6 will be described in Secs. 6.3.3 and 6.4.

3.4 Electroweak Penguin Operators

In spite of this rather moderate interest in our work in 1991 I was convinced that we should continue our project. The next step was to extend our calculation of $\hat{\gamma}_s^{(1)}$ in (2.37) to electroweak penguin operators Q_7, Q_8, Q_9 and Q_{10} and to calculate the ten dimensional two-loop anomalous dimension matrix $\hat{\gamma}_{se}^{(1)}$ that is necessary for the inclusion of the electroweak penguin operators at the NLO level with the goal to calculate the CP-violating ratio ε'/ε . Moreover, we wanted to write up the details of all these calculations. We have not done this in [94].

Unfortunately, Peter told me that he would only be involved in the calculation of $\hat{\gamma}_s^{(1)}$ as he was again very much involved in the collaboration with Martin Lüscher. Thus only Matthias, Markus and me were involved in the $\hat{\gamma}_{se}^{(1)}$ project that amounted in particular to the calculation of the two-loop diagrams in Fig. 5. Having all the machinery at hand we performed both calculations during the fall of 1991 and the winter 1991/1992 so that in March 1992 we had $\hat{\gamma}_s^{(1)}$ and $\hat{\gamma}_{es}^{(1)}$ including all the ten operators in the NDR and HV schemes. Moreover we calculated $\mathcal{O}(\alpha)$ corrections to the Wilson coefficients at $\mu = M_W$, an ingredient of the NLO analysis, that is necessary to remove the renormalization scheme dependence from the decay amplitudes.

Unfortunately, there was a problem with our result for $\hat{\gamma}_{es}^{(1)}$. While the $[\hat{\gamma}_s^{(1)}]_{NDR}$ and $[\hat{\gamma}_s^{(1)}]_{HV}$ were compatible with each other, $[\hat{\gamma}_{es}^{(1)}]_{NDR}$ and $[\hat{\gamma}_{es}^{(1)}]_{HV}$ were not. That is $[\hat{\gamma}_{es}^{(1)}]_{HV}$ obtained by the direct two-loop calculation differed by a small amount from $[\hat{\gamma}_{es}^{(1)}]_{HV}$ found from $[\hat{\gamma}_{es}^{(1)}]_{NDR}$ by means of a formula analogous to (2.39) that is given in [101].

We spent some time in order to clarify this discrepancy but after a few weeks we made a pause in our search for the error. I think we were simply exhausted. Moreover everyone was

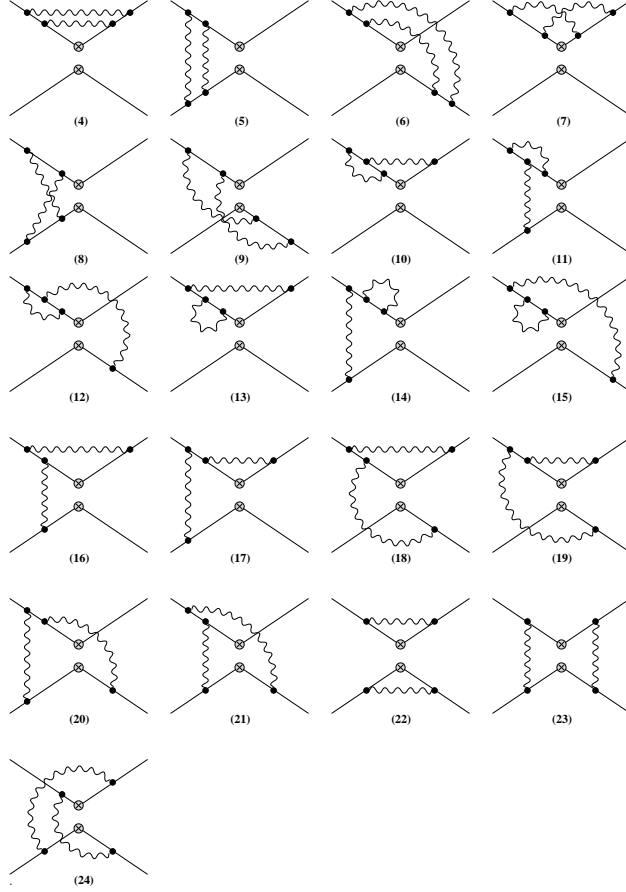


Figure 5: Two-loop current-current diagrams contributing to $\hat{\gamma}_{se}^{(1)}$. The wavy lines denote gluons or photons. The 4-vertices “ $\otimes \otimes$ ” denote standard operator insertions. Possible left-right or up-down reflected diagrams are not shown.

involved simultaneously in other projects: Peter with Martin Lüscher, Matthias and Markus with writing up their paper on ”TRACER”, the program, written in Mathematica, for Dirac algebra in $D \neq 4$ for NDR and HV schemes [110] and in my case in addition to being the head of the theory institute at our university and finishing a review article with Michaela Harlander, I started a new NLO climb: in the spring of 1992 our NLO club got the *fifth* member, a very important one, namely my new PhD student Gerhard Buchalla with whom I planned to attack at the NLO level all rare semi-leptonic K and B decays dominated by Z^0 -penguins. More about this in section 5.

Fortunately before making a pause in our climb we decided to write up the two papers, one including Peter on $\hat{\gamma}_s^{(1)}$ that did not have any problems and the second one without him on $\hat{\gamma}_{es}^{(1)}$ that had the problem mentioned above. Thus already in April 1992 our papers were essentially finished but before we could show them to the public we had still to solve the remaining problem

in the second paper.

From the beginning I was fully confident that our results in the NDR scheme were correct. The calculations were simpler than in the HV scheme and I was able to make several tests that all worked. As the calculations of $\Delta\hat{r}$ matrices relating the NDR and HV schemes is a one loop affair I was also confident that our results in the HV scheme obtained from the NDR scheme by means of a relation similar to (2.39) were also correct. However, the game of making a given NLO calculation in various schemes and checking the compatibility of the results started by Peter and myself three years before somehow fooled us and we did not send the papers for publication although we had all results already in April 1992. In the language of mountain climbing it is afterall irrelevant whether the first climb of a summit was done using the NDR “climbing style” or HV one.

Fortunately, the Rome group did not present their results at the 1992 summer conferences and consequently we were still in the game. Moreover, Matthias became the CERN fellow and could inform us in the first days of November 1992 that Guido Martinelli will give a seminar on ε'/ε beyond leading logarithms four weeks later. It was time to be active again. Feeling like colonel Hunt before the final attack to conquer the Mount Everest summit I convinced my collaborators to send out the paper on $\hat{\gamma}_s^{(1)}$ in the existing form and to present the details of the calculation of $\hat{\gamma}_{es}^{(1)}$ in the second paper only in the NDR scheme making the shift $\Delta\hat{r}$ to obtain it in the HV scheme. Our two papers [95, 101] appeared in the second half of November 1992, roughly two weeks before Guido’s CERN talk and three weeks before the Rome group sent out their letter to the Los Alamos archive [97].

To our delight the Rome team consisting of Marco Ciuchini, Enrico Franco, Guido Martinelli and Laura Reina agreed with our results on the anomalous dimension matrices in both NDR and HV schemes but to our surprise they did not present any details of their calculations of these matrices. Instead they presented their analysis of ε'/ε including NLO QCD and QED corrections. Thus although the Munich team has published as the first group all ingredients of a NLO analysis of $\Delta F = 1$ processes: two-loop anomalous dimensions and the Wilson coefficients at $\mu = M_W$, the Rome group was the first to present a NLO analysis of ε'/ε that included both QCD and electroweak penguin contributions.

Our NLO analysis of ε'/ε appeared in March 1993 in a very long paper (112 pages) that fortunately was accepted in this form in Nucl. Phys. B. [96]. We have presented there very explicit formulae for the Wilson coefficients of all operators and studied also the μ dependence of the hadronic B_i parameters. Moreover, we have proposed a method for extracting some of these parameters from the CP-conserving amplitudes. In this manner we could incorporate the $\Delta I = 1/2$ rule into the analysis of ε'/ε . The results of this analysis have been used by us in the 1990’s to gradually develop an approximate but rather accurate formula for ε'/ε that depends on three parameters R_6 , R_8 and Ω_{IB} with the first two representing the relevant matrix element of Q_6 and Q_8 operators and the last one summarizing isospin breaking corrections [117, 118].

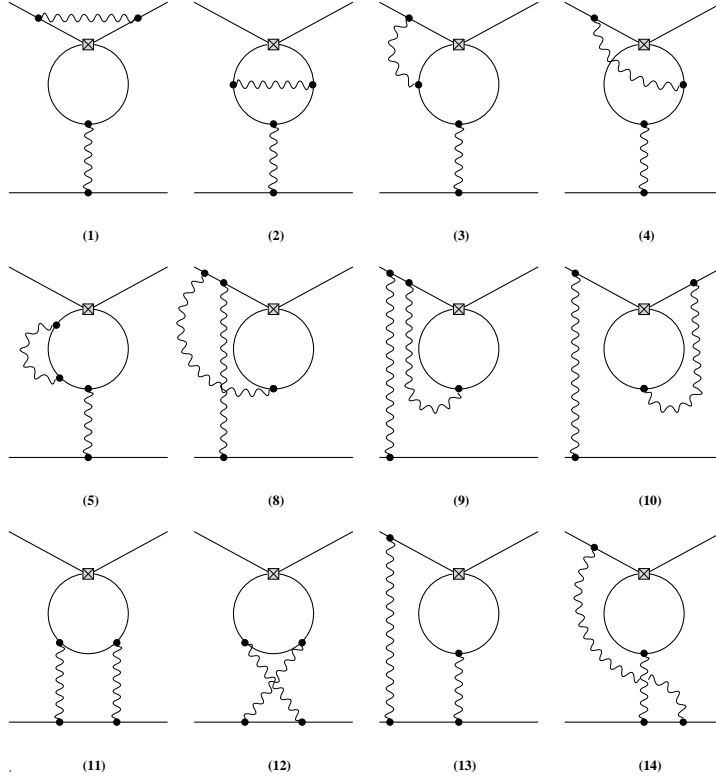


Figure 6: Two-loop penguin diagrams contributing to $\hat{\gamma}_{se}^{(1)}$. The wavy lines denote gluons or photons with each diagram containing one gluon and one photon. Square-vertices stand for two types of penguin insertions. Possible left-right reflected diagrams are not shown.

The last update of this formula can be found in [119].

As the formulae presented by the Rome group were not as explicit as ours, an analytic comparison between their final results and ours was not possible but numerically the agreement was within a few percent.

In our long paper [96] on ε'/ε we have also presented for the first time NLO Wilson coefficients of the operators relevant for non-leptonic $\Delta B = 1$ decays. Amusingly the interest of particle physics community in ε'/ε , in the middle of 1990's, was rather moderate and our ε'/ε paper of 1993 was cited in this period mainly for our B physics results.

One month after our final paper of this period the Rome group presented in great details their calculation of two-loop anomalous dimensions concentrating on the HV scheme [98]. This helped us to identify the error in our direct calculation of $[\hat{\gamma}_{es}^{(1)}]_{HV}$. Their result for this matrix indeed agreed with ours obtained in our paper indirectly through $[\hat{\gamma}_{es}^{(1)}]_{NDR}$.

The most important result of this Munich-Rome competition that lasted for several years is the agreement on the two-loop anomalous dimensions of the operators Q_i ($i = 1, \dots, 10$) that have

been used by the flavour community since then. While they were calculated with the aim to find ε'/ε at the NLO level, they play also essential role in all $\Delta F = 1$ transitions, not only the non-leptonic ones. Thus they also enter the NLO calculations of $B \rightarrow X_s \gamma$, $B \rightarrow X_s e^+ e^-$ and even $K \rightarrow \pi \nu \bar{\nu}$ decays. This is why in planning the grand expedition in 1988 it was essential to calculate these anomalous dimensions first.

In March 1993 I gave a seminar on our ε'/ε analysis in the theory group at CERN. I was approached there by two young physicists whom I did not meet before. It was Marco Ciuchini and Enrico Franco who came from Rome on a night train for a day to CERN to listen to my talk. I was really impressed that our competition went so far. However, it turned out that this friendly competition did not end in 1993.

3.5 More Operators

In 1997 an Italian group consisting of six climbers [30], including three from the expedition of early 1990's calculated two-loop anomalous dimensions of a set of operators relevant at NLO for $\Delta\Gamma_{s,d}$ in $B_{s,d}^0 - \bar{B}_{s,d}^0$ mixings and in particular for $\Delta F = 1$ and $\Delta F = 2$ non-leptonic decays and transitions in the extensions of the SM, like MSSM at large $\tan\beta$, multi-Higgs models and generally models that include in addition to left-handed currents also right-handed currents and scalar currents. These operators are given in the operator basis of [29] in (2.15) and (2.16).

I was a bit surprised that we have not looked at these operators, except for the first one, in Munich before. Otherwise we would calculate their anomalous dimensions already in 1993. The whole machinery developed by us at the beginning of the 1990's could be used here. Knowing this, there was essentially no point in repeating this new Italian calculation. The results given in [30] were bound to be correct. However, one result presented in this paper and even pointed out in the abstract made me interested in looking at this analysis closer.

The Italian group calculated the two-loop anomalous dimensions of the operators in (2.15) and (2.16) in the so-called RI scheme that is apparently useful for lattice calculations and in the NDR scheme. The calculation in the RI scheme was entirely new. The calculation for the operators $Q_{1,2}^{LR}$ in the NDR scheme was really not new as the anomalous dimensions of these operators can be directly obtained from our earlier calculations of QCD penguin operators. What was new in the NDR scheme were the two-loop anomalous dimensions of the operators $Q_{1,2}^{SLL}$. Here the Italian group found a surprising result: the analogs of Q_+ and Q_- operators in (3.8) mixed in this sector under renormalization. Their two-loop anomalous dimension matrix was non-diagonal and the results for the NDR presented in the appendix of this paper looked very complicated. If this was indeed true, the NDR scheme as defined by Peter and myself in [90] would not be an elegant scheme.

Also Mikolaj Misiak, who joined MNLC in 1995, was interested in this result. Together with Jörg Urban, a PostDoc in my group, we decided to look at the Italian paper closer. I should have noticed it right away, but it was Mikolaj who reminded me of the ‘‘Greek story’’

of missing evanescent operators in the case of Q_1 and Q_2 and the resulting mixing of Q_+ and Q_- found by Peter and myself in 1989. This time the Fierz vanishing evanescent operators were involved. I have discussed them already in section 3.1. We soon suspected that Italian masters were performing Fierz transformations in a $D \neq 4$ calculation without including the evanescent operators in question. We have all three independently performed the calculation of the anomalous dimensions of this sector now including the Fierz vanishing evanescent operators obtaining the result in the NDR scheme that was much simpler than the one of the Italian group, in particular the non-diagonal entries disappeared as they did in 1989 in the case of the current-current operators (Q_+, Q_-).

Mikolaj and Jörg confirmed the RI calculation of [30] in an arbitrary covariant gauge and found the matrices $\Delta\hat{r}$ relating the RI scheme and the NDR scheme. In this manner we could also find the compatibility of the RI result in [30] and our NDR result but the one loop matrices connecting these two schemes were clearly different from those given in [30].

Unfortunately, our Italian colleagues did not agree with our interpretation of the strange form of their result but they admitted that the renormalization scheme they used was not the standard NDR scheme of [90] and that our result was more elegant and phenomenologically more useful. In summary: the first NLO climb in the RI scheme related to the operators (2.15) and (2.16) should be credited to the Italian group while in the NDR scheme to us.

The operators in (2.15) and (2.16) do not constitute the full set of six-dimensional four quark operators contributing to $\Delta F = 1$ processes. In addition to QCD penguins and electroweak penguins of the SM there are other penguin operators. In our paper [29] we have therefore generalized our analysis of two-loop anomalous dimensions to the full set of $\Delta F = 1$ four-quark operators. These results are much less known but should be useful in the extensions of the SM one day.

3.6 QCD Corrections to Semi-leptonic B Decays

This is probably a good place to summarize the QCD corrections to semi-leptonic B decays that are necessary for an accurate determination of the CKM elements $|V_{cb}|$ and $|V_{ub}|$. As I did not take part in these calculations I consulted a very prominent member of the MNLC club, Paolo Gambino, who contributed in an important manner to these calculations. As we will soon see another member of our club, Andrzej Czarnecki, who will enter the scene later on, also made important contributions to this field. A nice summary of this topic as far as $b \rightarrow cl\nu$ is concerned can be found in [120].

In what follows I will list papers where only the purely perturbative corrections to semileptonic B decays have been computed leaving the summary of power corrections to other reviewers.

3.6.1 NLO Corrections to Inclusive $B \rightarrow X_c l \nu$ Decays

The pioneering first steps in this field can be found in [121]. The first complete analytic calculations, not only rate but also a few differential distributions, have been performed much later by Andrzej Czarnecki, Marek Jezabek and Hans Kühn [122]. In this context also the corrections to the rate in a compact form found by Yossi Nir should be mentioned [123].

Next corrections to moments of hadronic spectra which used the results of previous calculations can be found in [124], while full triple differential distribution at $\mathcal{O}(\alpha_s)$ that are necessary for realistic experiments have been obtained in [125] and in particular in [126].

3.6.2 NNLO Corrections to Inclusive $B \rightarrow X_c l \nu$ Decays

First BLM-NNLO corrections to the rate have been obtained in [127] and in particular in [128]. The BLM- $\mathcal{O}(\alpha_s^2 \beta_0)$ corrections to lepton spectrum can be found in [129] and to triple differential distributions in [126].

Next non-BLM two-loop corrections (analytic, zero cut) have been obtained in [130] and in the numerical form but with realistic cuts in [131]. Finally complete two-loop corrections at specific kinematic points (zero recoil, which is important for the determination of $|V_{cb}|$ from $B \rightarrow D^{(*)} l \nu$) can be found in [132].

3.6.3 NLO and NNLO Corrections to Inclusive $B \rightarrow X_u l \nu_l$ Decays

NNLO complete calculation of the width has been performed in [133]. NLO and BLM-NLO full triple differential distributions have been calculated in [134] and [135], respectively. Leading shape functions and resummation in $B \rightarrow X_u l \nu_l$ has been done in [136] and non-leading shape functions in [137]. The most recent calculations dealing with this topic are [138, 139].

4 $\Delta S = 2$, $\Delta B = 2$ and $\Delta B = 0$ Transitions

4.1 Effective Hamiltonians for $\Delta F = 2$ Transitions

Let us begin this section by recalling the effective Hamiltonians for $\Delta S = 2$ and $\Delta B = 2$ transitions in the SM. We have first [140, 141, 142]

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{\Delta S=2} &= \frac{G_F^2}{16\pi^2} M_W^2 [\lambda_c^2 \eta_1 S_0(x_c) + \lambda_t^2 \eta_2 S_0(x_t) + 2\lambda_c \lambda_t \eta_3 S_0(x_c, x_t)] \times \\ &\times [\alpha_s^{(3)}(\mu)]^{-2/9} \left[1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right] Q(\Delta S = 2) + h.c. \end{aligned} \quad (4.1)$$

where $\lambda_i = V_{is}^* V_{id}$. Here $\mu < \mu_c = \mathcal{O}(m_c)$. In (4.1), the relevant operator

$$Q(\Delta S = 2) = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A} \quad (4.2)$$

is multiplied by the corresponding coefficient function. This function is decomposed into a charm-, a top- and a mixed charm-top contribution with $S_0(x_i)$ and $S_0(x_i, x_j)$ being one-loop box functions in the SM.

Short-distance QCD effects are described through the correction factors η_1, η_2, η_3 and the explicitly α_s -dependent terms in the last line of (4.1). This factor allows to introduce the renormalization group invariant parameter \hat{B}_K by

$$\hat{B}_K = B_K(\mu) \left[\alpha_s^{(3)}(\mu) \right]^{-2/9} \left[1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right], \quad (4.3)$$

$$\langle \bar{K}^0 | (\bar{s}d)_{V-A} (\bar{s}d)_{V-A} | K^0 \rangle \equiv \frac{8}{3} B_K(\mu) F_K^2 m_K^2. \quad (4.4)$$

The corresponding Hamiltonian for $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixing has similar structure but it is simpler as only the top contribution matters. We have for $B_q^0 - \bar{B}_q^0$ mixing

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{\Delta B=2} &= \frac{G_F^2}{16\pi^2} M_W^2 (V_{tb}^* V_{tq})^2 \eta_B S_0(x_t) \times \\ &\times \left[\alpha_s^{(5)}(\mu_b) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi} J_5 \right] Q^q(\Delta B = 2) + h.c. \end{aligned} \quad (4.5)$$

Here $\mu_b = \mathcal{O}(m_b)$,

$$Q^q(\Delta B = 2) = (\bar{b}q)_{V-A} (\bar{b}q)_{V-A}, \quad q = d, s. \quad (4.6)$$

The renormalization group invariant parameters \hat{B}_q are defined by

$$\hat{B}_{B_q} = B_{B_q}(\mu) \left[\alpha_s^{(5)}(\mu) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu)}{4\pi} J_5 \right] \quad (4.7)$$

$$\langle \bar{B}_q^0 | (\bar{b}q)_{V-A} (\bar{b}q)_{V-A} | B_q^0 \rangle \equiv \frac{8}{3} B_{B_q}(\mu) F_{B_q}^2 m_{B_q}^2, \quad (4.8)$$

where F_{B_q} is the B_q -meson decay constant.

We are now ready to discuss the history of the NLO QCD calculations of

$$\eta_1 \equiv \eta_{cc}, \quad \eta_2 \equiv \eta_{tt}, \quad \eta_3 \equiv \eta_{ct}, \quad \eta_B. \quad (4.9)$$

4.2 The Top Quark Contributions

In the fall of 1989 Matthias Jamin joined my group becoming the third member of the MNLC. Instead of continuing our NLO calculations for $\Delta F = 1$ transitions we decided to calculate the NLO QCD corrections to top quark contributions to the effective Hamiltonians for $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings and $K^0 - \bar{K}^0$ mixing. We were not the first to do this climb but the first two attempts were unsuccessful. The calculations were plainly wrong with the results for the Wilson

coefficients exhibiting infrared regulator dependence that was a consequence of an incorrect matching of the full and effective theories. Moreover these calculations did not include the two-loop anomalous dimension of the operator $Q(\Delta F = 2)$. As I have a high respect for the leaders of these two expeditions, that otherwise had significant contributions to our field, I prefer not to refer to these papers.

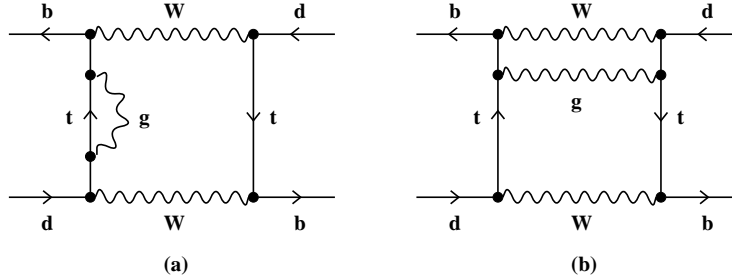


Figure 7: Examples of two-loop diagrams contributing to $B_d^0 - \bar{B}_d^0$ mixing.

Our new NLO project [142] involved two parts. One very easy, the other one rather difficult. The first one was the calculation of the two-loop anomalous dimension of the relevant operator $Q(\Delta F = 2)$. It took no time as the anomalous dimension of $Q(\Delta F = 2)$ is simply equal to γ_+ , the anomalous dimension of Q_+ in (3.8) calculated at the two-loop level by Peter and myself half a year before. The second part was much more time consuming. It involved the calculation of $\mathcal{O}(\alpha_s)$ QCD corrections to the box diagrams involving internal W^\pm bosons, the Goldstone bosons ϕ^\pm and the top quark exchanges. This is a two-loop calculation in a full theory with massive heavy particles. Two examples of contributing diagrams are shown in Fig. 7. The remaining diagrams can be found in Fig. 2 of [142]. There are 8 classes of diagrams in total shown in this figure from which the remaining diagrams can easily be obtained.

In order to extract the relevant Wilson coefficient of the operator $Q(\Delta F = 2)$ from this calculation the proper matching to the effective theory has to be made. This requires the calculation of the $\mathcal{O}(\alpha_s)$ corrections to the matrix element of this operator between the external quark states. The two-loop calculation of the $\mathcal{O}(\alpha_s)$ corrections to the box diagrams involves infrared divergences. We decided to set the external momenta to zero and regulate the infrared divergences by the masses of the external quarks. Working off-shell introduces necessarily gauge dependence (gluon propagator) in the final result. The final result of the box diagram calculation was therefore gauge and infrared regulator dependent. These dependences certainly did not belong to the Wilson coefficient of the operator $Q(\Delta F = 2)$ but to its matrix element and were removed in the process of matching of the full theory to the effective theory with the latter exhibiting precisely the same gauge and infrared regulator dependences.

The main results of our paper were the values of the QCD factors η_B and η_2 for $B_{d,s}^0 - \bar{B}_{d,s}^0$

and $K^0 - \bar{K}^0$ Hamiltonians, respectively:

$$\eta_B = 0.55 \pm 0.01, \quad \eta_2 = 0.57 \pm 0.01 . \quad (4.10)$$

Our result for η_B has been confirmed several years later in [64].

These QCD factors have been used in the last 20 years frequently in the literature. While using these particular values one should remember that they should be used in conjunction with the renormalization group invariant parameters \hat{B}_K and $\hat{B}_{s,d}$ in (4.3) and (4.7), respectively. These parameters have been introduced at the LO already in 1983 by Wojtek Slominski, Herbert Steger, and myself [140] and have been generalized to NLO in the present calculation. I do not remember who is the father of the B_K parameter without the “hat”, but I think it is John Donoghue.

This takes care of the μ dependence at the lower end of the RG evolution. η_i resulting from the calculation of $C(\mu)$ and \hat{B}_i representing the matrix element $\langle Q(\mu) \rangle$ are so defined that they separately do not depend on μ . However, η_i depend on the scale $\mu_W = \mathcal{O}(M_W)$ at which the matching between the full and the effective theory is made. This dependence is cancelled by the μ_W dependence of the initial conditions $C(\mu_W)$ and this cancellation is only meaningful at the NLO. More importantly the η_i are so defined that they multiply the leading order box functions and consequently they include the $\mathcal{O}(\alpha_s)$ corrections to the box diagrams that Peter, Matthias and me calculated. Consequently they depend also on the scale μ_t at which the running top quark mass $m_t(\mu_t)$ used in the calculation is defined. This μ_t dependence of η_i is important as it cancels up to higher order corrections the μ_t dependence of the Inami-Lim function $S_0(x_t(\mu_t))$ that remained uncompensated at LO. The values in (4.10) correspond to $\mu_t = m_t$. This turns out to be a convenient choice as with $\mu_t = m_t$, the QCD factors are practically independent of the actual measured value of $m_t(m_t)$. The leading logarithm multiplying a large anomalous dimension of the mass operator vanishes at this scale. I mention this issue again because it is important.

To my knowledge the issue of the μ_t uncertainty in $\Delta F = 2$ transitions in the LO and its reduction through the inclusion of NLO QCD corrections, was for the first time addressed in [142]. I have described it here because it enters all the calculations presented below and our paper completed in the Spring of 1990 can be considered as the prototype of analogous two-loop calculations of the Wilson coefficients for any FCNC process that is sensitive to the top quark mass or any other heavy particle with colour, for instance squarks in the MSSM.

I should emphasize that our calculation of NLO corrections to box diagrams with top quark exchanges performed in 1989-1990 had much simpler structure than the LO calculation of renormalization group effects done by Fred Gilman and Mark Wise already in 1983 [46]. The reason is that in 1983 the typical values of m_t considered in the literature were substantially lower than M_W and Fred and Mark in their 1983 LO calculation had to integrate out first W^\pm and subsequently the top quark at much lower scales. In the range $m_t \leq \mu \leq M_W$ they had to deal

with QCD corrections to bilocal operators originating in the contraction of the W^\pm propagator in a box diagram to a point but leaving the top quark propagator as it is. The renormalization and the calculation of QCD corrections to these bilocal structures are rather involved even in the LO. Fortunately in 1989 it was already known that $M_W \leq m_t \leq 200 \text{ GeV}$. Consequently W^\pm and the top quark could be integrated out simultaneously generating a local operator from the beginning and the QCD renormalization of the bilocal structures at NLO, a formidable task, could be avoided in our calculation.

Table 3: NLO and NNLO Calculations for $\Delta F = 2$ and $\Delta F = 0$ Transitions

Decay	NLO	NNLO
η_1	[143]	[146]
η_2, η_B	[142, 64]	
η_3	[144, 145]	
ADMs BSM	[30, 29]	
$\Delta\Gamma_{B_s}$	[150, 151, 152, 154]	
$\Delta\Gamma_{B_d}$	[152, 153, 154]	
$\Delta F = 2$ Tree-Level	[156]	

4.3 Charm and Top-Charm Contributions

In the case of box diagrams with two charm quark exchanges there is no way out. One has to face the bilocal structures at NLO because the simultaneous integration of the charm quark and W^\pm would lead to $\ln M_W/m_c$ terms and consequently to the breakdown of perturbation theory. This rather difficult project was assigned in 1992 to my PhD student Stefan Herrlich. Stefan was a very good student and possibly he would succeed this climb by himself but fortunately for him and for the project, he was joined early 1993 by Ulrich Nierste. Ulrich got his diploma in Würzburg working with the loop masters like Manfred Böhm and Ansgar Denner and consequently he was fit for this difficult climb almost immediately after his arrival in Munich. In fact Ulrich was soon leading this important climb and developed to one of the most prominent members of the MNLC.

The calculations of $\eta_1 = \eta_{cc}$ and $\eta_3 = \eta_{ct}$ QCD factors at NLO were completed in 1993 [143] and 1995 [144], respectively and the full analysis of $\Delta S = 2$ Hamiltonian at the NLO level could be performed soon after [145]. These were truly heroic climbs that were not repeated by anybody until 2010, when two younger members of the MNLC performed the NNLO calculation of η_3 : one of my many physics sons, Martin Gorbahn, a big star these days in multiloop calculations and my physics grandson, a PhD student of Uli Nierste, Joachim Brod [146]. It looks like the η_i factors still remain in the possession of our physics family.

Even Guido Martinelli was impressed by the Herrlich-Nierste calculations and he told me this at least three times at different occasions. The values for η_1 and η_3 enter the analysis of the CP-violating parameter ε_K and are relevant ingredients of any analysis of the unitarity triangle, in particular after the lattice value for \hat{B}_K [113] and the estimate of long distance effects in ε_K improved recently [148, 149]. I was through all these years convinced, knowing Stefan and Ulrich, that the values for η_1 and η_3 found by them were correct and the recent calculation of η_3 by Brod and Gorbahn confirmed my expectations. It would be good to know the NNLO value of η_1 as well. Even with the technology of the present decade these NNLO calculations are heroic efforts.

In fact Brod and Gorbahn completed this calculation few months after V2 of this review appeared [147]. The result is a bit disappointing as the NNLO calculation did not reduce the uncertainties in the charm part. Possibly with a better matching to the lattice calculations this can be achieved one day.

4.4 $\Delta\Gamma_s$ and $\Delta\Gamma_d$ at NLO

In the winter 1997/1998 I have been asked by Gerhard Buchalla, whether I would be interested in joining him, Martin Beneke, Christoph Greub, Alexander Lenz and Uli Nierste in the calculation of NLO QCD corrections to the life-time difference or equivalently $\Delta\Gamma_s$ in the $B_s^0 - \bar{B}_s^0$ system. At first I found it an interesting idea. Afterall $\Delta\Gamma_s$ is much larger than $\Delta\Gamma_d$ and working with my physics sons Alex and Uli, my physics stepson Martin and the Swiss master Christoph for the first time would be a real fun. In 1984 I studied $\Delta\Gamma_d$ at LO with Slominski and Steger [141] and in addition after nine years of NLO climbing I was well prepared for this new expedition. Yet, this winter I was busy with writing up my Les Houches lectures and other project and I did not join Gerhard et al. Two papers in 1998 and 2002 [150, 153, 154] as well as numerous phenomenological analyses of Lenz and Nierste in the last decade resulted from this project [155]. Equally important this topic was the subject of the PhD thesis of Alexander Lenz.

Interestingly, also the younger generation of Rome masters got involved in these efforts [151, 152], in particular Cecilia Tarantino, who became in 2006 my close one-loop collaborator within the Littlest Higgs Model with T-parity. I hope one day somebody will report on this competition, although as most of the authors are at least twenty years younger than me, it will still take some time. The results of these four papers played already an important role in the analyses of the Tevatron data and will certainly be very important for the LHCb.

4.5 NLO QCD Corrections to Tree-Level $\Delta F = 2$ Processes

NLO QCD corrections to box diagrams are rather involved and it appears a bit premature to calculate them for extensions of the SM. Therefore, between 2006 and 2011, when I exclusively studied extensions of the SM, I decoupled from calculations of NLO QCD corrections. However,

in the fall of 2011 I noticed that for $\Delta F = 2$ processes mediated at tree-level by colourless neutral gauge boson and neutral Higgs exchanges the matching conditions at NLO require only one-loop calculations and can be done model independently without too much effort. Somehow QCD experts including myself did not notice it before.

In April 2011 Jennifer Girrbach, PhD student of Ulrich Nierste and consequently my granddaughter in physics, joined my group. Jennifer had no experience in QCD calculations but I thought it would be fun to perform this climb with her and to teach her this field. Moreover, she was one of the stars of Ulrich's group and I was sure that we will reach this summit together. Indeed Jennifer learned the QCD technology in a short time and performed all necessary calculations independently of me. We published our results on $\Delta F = 2$ transitions already in January 2012 [156]. Since then we could use these results for concrete models with Z' tree-level exchanges. We could subsequently extend this calculation to non-leptonic $\Delta F = 1$ processes mediated by colourless neutral gauge boson and neutral Higgs exchanges [157].

5 Rare K and B Decays

5.1 Effective Hamiltonians for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$

These decays have been with us already for more than twenty years and the world of particle physics is waiting for their precise measurements. A review published in 2008 [158] summarizes the status of two years ago. More recent developments are presented below and in [84].

The effective Hamiltonian for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ can be written as

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} \sum_{l=e,\mu,\tau} \left(V_{cs}^* V_{cd} X^l(x_c) + V_{ts}^* V_{td} X(x_t) \right) (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A}. \quad (5.1)$$

The index $l=e, \mu, \tau$ denotes the lepton flavour. The dependence on the charged lepton mass resulting from the box-graph is negligible for the top contribution. In the charm sector this is the case only for the electron and the muon but not for the τ -lepton.

The function $X(x_t)$ relevant for the top part is given by

$$X(x_t) = X_0(x_t) + \frac{\alpha_s}{4\pi} X_1(x_t) \quad (5.2)$$

with the leading contribution $X_0(x)$ resulting from Z penguin diagrams and box-diagrams and $X_1(x_t)$ denoting QCD corrections to these diagrams that will be discussed below.

In the case of charm contributions it is useful to define the parameter

$$P_c(X) = \frac{1}{\lambda^4} \left(\frac{2}{3} X^e(x_c) + \frac{1}{3} X^\tau(x_c) \right), \quad (5.3)$$

with $\lambda = |V_{us}|$ being the Wolfenstein parameter ($\lambda \approx 0.225$).

Keeping terms to first order in α_s , the perturbative expansion of $P_c(X)$ has the following general structure

$$P_c(X) = \frac{4\pi}{\alpha_s(\mu_c)} P_c^{(0)}(X) + P_c^{(1)}(X) + \frac{\alpha_s(\mu_c)}{4\pi} P_c^{(2)}(X), \quad (5.4)$$

In the case of the decay $K_L \rightarrow \pi^0 \nu \bar{\nu}$ only the top function $X(x_t)$ matters. Similarly in the case of $B \rightarrow X_s \nu \bar{\nu}$ only this function matters.

5.2 Effective Hamiltonians for $K_L \rightarrow \mu^+ \mu^-$ and $B_{s,d} \rightarrow \mu^+ \mu^-$

In the case of $K_L \rightarrow \mu^+ \mu^-$ only subleading short distance (SD) part can be computed. The analysis of this part proceeds in essentially the same manner as for $K^+ \rightarrow \pi^+ \nu \bar{\nu}$. The only difference is introduced through the reversed lepton line in the box contribution. In particular there is no lepton mass dependence, since only massless neutrinos appear as virtual leptons in the box diagram.

The effective hamiltonian in next-to-leading order can be written as follows:

$$\mathcal{H}_{eff}(K_L \rightarrow \mu^+ \mu^-) = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} (V_{cs}^* V_{cd} Y(x_c) + V_{ts}^* V_{td} Y(x_t)) (\bar{s}d)_{V-A} (\bar{\mu}\mu)_{V-A} + h.c. \quad (5.5)$$

The function $Y(x)$ is given by

$$Y(x_t) = Y_0(x_t) + \frac{\alpha_s}{4\pi} Y_1(x_t), \quad (5.6)$$

and

$$P_c(Y) = \frac{Y(x_c)}{\lambda^4} \quad (5.7)$$

has an expansion similar to $P_c(X)$ in (5.4).

Only the function $Y(x_t)$ is relevant for $B_{s,d} \rightarrow \mu^+ \mu^-$ for which the effective Hamiltonian reads

$$\mathcal{H}_{eff}(B_s \rightarrow \mu^+ \mu^-) = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} V_{tb}^* V_{ts} Y(x_t) (\bar{b}s)_{V-A} (\bar{l}l)_{V-A} + h.c. \quad (5.8)$$

with s replaced by d in the case of $B_d \rightarrow l^+ l^-$.

We give here only the diagrams contributing to $Y_0(x_t)$ to emphasize that only the box diagrams have to be calculated relative to the $\nu \bar{\nu}$ case as the direction of the internal lepton line differs in this case (compare with Fig. 9).

We are now ready to discuss NLO and NNLO QCD corrections to these decays.

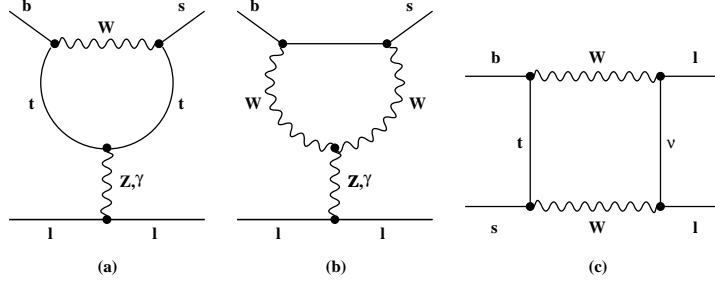


Figure 8: One loop diagrams contributing to rare decays with charged leptons in the final state.

5.3 NLO and NNLO QCD Calculations

The story of NLO QCD corrections to rare K and B decays begins in the fall of 1990 when I started the calculation of $\mathcal{O}(\alpha_s)$ corrections to the flavour changing Z^0 -penguin one-loop diagrams that dominate semi-leptonic rare decays like $K \rightarrow \pi\nu\bar{\nu}$, $B_{s,d} \rightarrow \mu^+\mu^-$ and $B \rightarrow X_{s,d}\nu\bar{\nu}$. The calculation involves 30 two-loop diagrams with three examples shown in Fig. 9. Most of them are free from infrared divergencies so that the external masses and momenta can be neglected. The infrared divergent diagrams can be regulated by non-vanishing external masses but a more elegant method is the dimensional regularization.

I have started this climb by myself but after roughly twenty two-loop diagrams I stopped. I found this solo climb doable but I was already involved in the calculation of two-loop anomalous dimensions of penguin operators described previously and moreover it is always safer to have a partner in the climbs of that sort. Fortunately, in contrast to the ordinary climbing, in situations like that there is no need to return to the base camp. With good notes the climb can be continued whenever one decides to do it.

In July 1991 Gerhard Buchalla returned from the military service, fit to begin his PhD studies and to join me in the Z^0 -penguin climb. However, as a warming-up I suggested to him a one-loop calculation: $\mathcal{O}(\alpha_s)$ corrections to non-leptonic c -quark decay at $\mu = \mathcal{O}(m_c)$ in the NDR and HV schemes that in 1981 was done by Altarelli et al. in the DRED scheme. Gerhard found the compatibility of his results with those of [5, 6], published them [106]^{||}, and started from the base camp to climb the Z^0 -penguin NLO QCD summit in the early summer of 1992. This clearly motivated me to continue my climb of 1990. Gerhard was one of my best PhD students ever and even if we were climbing separately, I knew that after reaching the summit, I would meet him there to compare my results with his.

As a CERN fellow from 1975 to 1977 I had the opportunity to talk to one of the old masters

^{||}Such calculations have been refined in the context of $b \rightarrow ccs$ [107, 108] with the participation of my assistant, Patricia Ball.

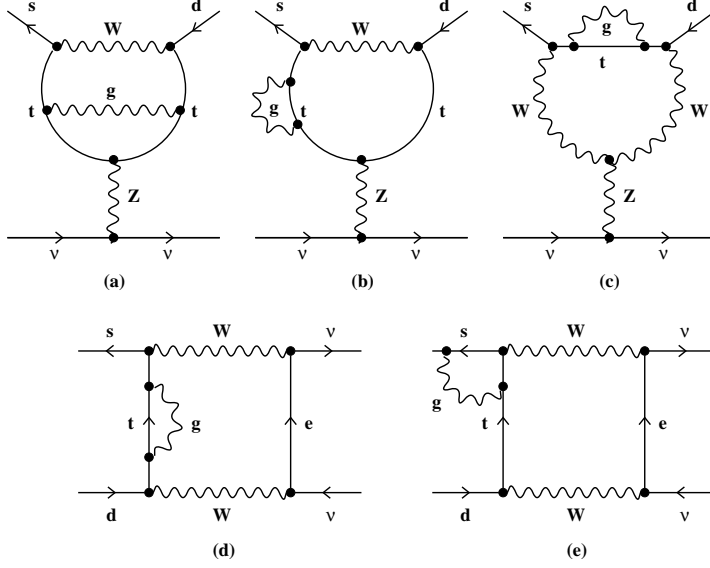


Figure 9: Examples of two-loop diagrams contributing to $X_1(x_t)$.

of higher order calculations of $g - 2$, A. Peterman. He told me that in order to be sure that a result of a lengthy multi-loop calculation is correct, the climbing partners, should have as little contact with each other as possible, comparing their results only at the end of the climb.

While in most calculations, I have done in the 1990's, I followed Peterman's advice as much as possible, the calculation of $\mathcal{O}(\alpha_s)$ corrections to Z^0 -penguin diagrams with Gerhard could be considered as a perfect example of such an approach. Our calculations were totally independent. I have no idea how he got the final result and the same applies to him with respect to my calculation. In the fall of 1992 we compared our results diagram by diagram reaching full agreement on all 30 diagrams except for one term in one diagram, that turned out to be a misprint in my notes. We were rather confident that our result was correct.

As a byproduct we could extract from our $\mathcal{O}(\alpha_s)$ corrections to the $Z\bar{b}s$ vertex the $\mathcal{O}(\alpha_s)$ corrections to the flavour conserving $Z^0 b\bar{b}$ penguin diagram in the large m_t limit, being in fact the first group that confirmed the results of [159] done in the context of electroweak precision studies three months earlier. In the first years after the appearance of our paper [48], it got most citations precisely for this additional calculation. After the discovery of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in 1997 things of course changed.

In the following paper [49], after calculating $\mathcal{O}(\alpha_s)$ QCD corrections to $\Delta F = 1$ box diagrams, we could finally present $\mathcal{O}(\alpha_s)$ corrections to all rare K and B decays dominated by internal top quark exchanges: $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_{s,d} \rightarrow \mu^+ \mu^-$ and $B \rightarrow X_{s,d} \nu \bar{\nu}$. In the case of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ we had still to calculate NLO QCD corrections to the internal charm contributions. This was the subject of our third and final paper of this period [51] which required the inclusion

of QCD corrections to the bilocal structures as the ones given in Fig. 10.

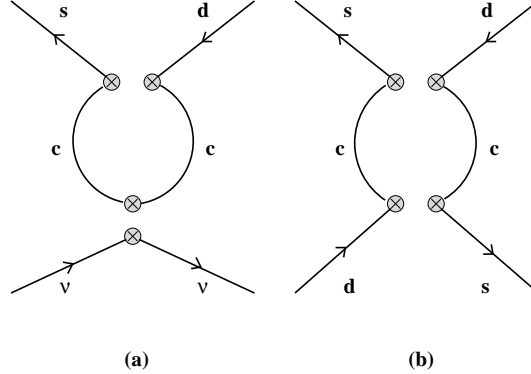


Figure 10: Bilocal Structures.

Our calculations of two first papers reduced the $\mathcal{O}(15\%)$ uncertainty in the branching ratios for $K_L \rightarrow \pi^0 \nu \bar{\nu}$, $B_{s,d} \rightarrow \mu^+ \mu^-$ and $B \rightarrow X_{s,d} \nu \bar{\nu}$ due to the choice of μ_t in $m_t(\mu_t)$ present in the LO calculations in [47] down to $\pm 1\%$. Amusingly the most important phenomenological contribution of the second paper is the realization that most, if not all, papers in the literature missed an overall factor of 2 in the branching ratio for $B_s \rightarrow \mu^+ \mu^-$. When I mentioned during an experimental discussion at Beauty 1995 in Oxford that the simulations for the LHCb for this decay should use a branching ratio by a factor of two higher, there was a real joy among my experimental colleagues. Afterall a branching ratio of 3.2×10^{-9} is easier to measure than 1.6×10^{-9} .

Concerning QCD calculations the most important achievement of these three papers is the the calculation of $P_c^{(1)}(X)$ in (5.4). This allowed the reduction of the uncertainty due to the choice of μ_c in $m_c(\mu_c)$ in the $P_c(X)$ from $\pm 26\%$ in the LO down to $\pm 10\%$. Ten years later this calculation has been extended to the NNLO level, the last term in (5.4), by Martin Gorbahn, Ulrich Nierste, Ulrich Haisch, still another multi-loop star among my PhD students, and myself [31], reducing the uncertainty in question down to $\pm 2\%$. This should be a very relevant improvement when data on $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ will become accurate. As in the most extensions of the SM, the charm contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ remains essentially unaffected by new physics, these results are also relevant for most extensions of the SM.

In the case of $K_L \rightarrow \mu^+ \mu^-$ our NLO calculation reduced the μ_c uncertainty in $P_c(Y)$ from $\pm 44\%$ present in the LO down to $\pm 22\%$. Ten years later this calculation has been extended to the NNLO level by Martin Gorbahn and Ulrich Haisch [52] reducing this uncertainty down to $\pm 7\%$. As the charm contribution is much less relevant in this decay than in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ is subject to non-perturbative uncertainties, this left over uncertainty is practically negligible. Finally, our calculations of NLO QCD corrections to $Y(x_t)$ basically eliminated μ_t uncertainty both in $K_L \rightarrow \mu^+ \mu^-$ and $B_{s,d} \rightarrow \mu^+ \mu^-$.

In September 1998, almost five years after the completion of our Z^0 -trilogy, that also included QCD corrections to the relevant $\Delta F = 1$ box diagrams, I spent one month in the CERN theory group developing a general parametrization for $B \rightarrow \pi K$ decays in collaboration with Robert Fleischer and rather decoupled from $K \rightarrow \pi \nu \bar{\nu}$ decays. Also Gerhard Buchalla was there. One day we received an e-mail from Mikolaj Misiak who in collaboration with Jörg Urban calculated $\mathcal{O}(\alpha_s)$ corrections to the top contributions to rare decays [160], the subject of our first two papers. Their result for Z^0 -penguin contributions obtained using the external masses of quarks as infrared regulator agreed with our result but the one for box diagrams disagreed with our calculation. The difference was phenomenologically irrelevant (1–2%) but there was a difference. I was already worried that I had to go again through my lengthy two-loop calculations of 1992 but fortunately we got a second e-mail. Mikolaj and Jörg did also the calculation of box diagrams regulating infrared divergences dimensionally, as we did five years before, confirming our results in the full theory. This was a true relief. However, they found that our very simple calculation in the effective theory did not include an evanescent operator related to the dimensional infrared regulator that had to be added in the case of the box part. After including it, the small difference between our results disappeared.

The issue of including the evanescent operators in this case has nothing to do with the cases discussed in section 3 and is rather sophisticated. It is discussed in details in [160, 161]. This example shows that the method of using dimensional regularization to regulate infrared divergences and not distinguishing this regulator from the dimensional regulator of ultraviolet divergences, although very elegant and correct, does not allow a good test of the final result. In this respect regulating the infrared divergences by external quark masses, as was done in our calculations of η_2 and η_B and also by Mikolaj and Jörg in their calculation of rare decays, is more difficult but safer.

Mikolaj and Jörg checked only our calculation of top quark contributions identifying the small difference mentioned above but somehow they were not interested in looking again at the internal charm contributions. This is what Gerhard and I did in [161] adding the small contribution of the evanescent operator to our old result. Our 1999 paper can be considered as a compendium of all expressions for the Wilson coefficients relevant for rare decays $K \rightarrow \pi \nu \bar{\nu}$, $B_{s,d} \rightarrow \mu^+ \mu^-$ and $B \rightarrow X_{s,d} \nu \bar{\nu}$ in the SM at the NLO level. The hard work has been done in our first three papers that are also summarized and discussed in our review [15] but the final fully correct expressions at the NLO level are given in our 1999 paper.

The NNLO QCD calculations for the charm component in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ [31] and in $K_L \rightarrow \mu^+ \mu^-$ [52] were of course much more involved and included several two-loop and in particular three loop diagrams in the bilocal operator sector. The result was the $P_c^{(2)}(X)$ in the case of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and the $P_c^{(2)}(Y)$ in the case of $K_L \rightarrow \mu^+ \mu^-$. I will not describe them here as they have been described in great detail in two papers on $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ I was involved in and in the paper by Gorbahn and Haisch [52] on $K_L \rightarrow \mu^+ \mu^-$. These calculations further reduced

the perturbative uncertainties in these decays to the level of $\pm 2\%$ so that the most important uncertainties in the corresponding branching ratios reside in the CKM element $|V_{cb}|$ that enters the branching ratios as $|V_{cb}|^4$.

In Table 4 we collect references to QCD calculations for rare K and B decays. A given entry means that full NLO or NNLO corrections to the decay in question have been calculated in the quoted paper.

Table 4: Rare K and B decays

Decay	NLO	NNLO
$K_L^0 \rightarrow \pi^0 \nu \bar{\nu}$, $B \rightarrow l^+ l^-$, $B \rightarrow X_s \nu \bar{\nu}$	[48, 49, 160, 161]	
$K^+ \rightarrow \pi^+ \nu \bar{\nu}$	[51, 161]	[31]
$K_L \rightarrow \mu^+ \mu^-$	[51, 161]	[52]
$K^+ \rightarrow \pi^+ \mu \bar{\mu}$	[165]	
EW to Charm in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$	[166]	
EW to Top in $K \rightarrow \pi \nu \bar{\nu}$	[56, 57]	

5.4 Two-loop Electroweak Contributions

The rare decays like $K \rightarrow \pi \nu \bar{\nu}$ are theoretically very clean as all non-perturbative effects investigated by a number of authors [162, 163, 164] have been found very small and definitely below any experimental sensitivity in the coming ten years. For this reason it is of interest to investigate also two-loop electroweak contributions to rare decays.

At first sight these contributions appear to be negligible. This however is not fully true for the following reason. The effective Hamiltonian for $K \rightarrow \pi \nu \bar{\nu}$ decays involves electroweak parameters like G_F , α and in particular $\sin^2 \theta_W$ that all depend on the renormalization scheme used in the usual electroweak precision studies. This dependence can only be reduced by including higher order electroweak corrections to the leading one-loop diagrams in rare decays. This means the calculation of two-loop electroweak diagrams. Consider for instance $\sin^2 \theta_W$. The Particle Data Group gives two values for this parameter: $(\sin^2 \theta_W)_{\overline{MS}} = 0.231$ and $(\sin^2 \theta_W)_{\text{eff}} = 0.224$, and of course there are other possibilities. As $Br(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ is inversely proportional to $\sin^4 \theta_W$ the two choices give two values for $Br(K^+ \rightarrow \pi^+ \nu \bar{\nu})$ that differ by 6%. This is clearly irrelevant today but I hope that the experimental data will improve in this decade to the extent that this difference will matter.

A calculation of two-loop electroweak effects is clearly a very difficult affair but fortunately Gerhard and me could remove the ambiguity in question approximately without doing this calculation at all [56]. We noticed that the calculations of similar effects in the context of electroweak precision studies contained sufficient information to find two-loop electroweak contributions to

$K \rightarrow \pi\nu\bar{\nu}$ in the large m_t limit without performing any loop calculations. Adding these contributions to our previous result reduced the ambiguity in question below 1%. Moreover, with the choice of $(\sin^2\theta_W)_{\overline{MS}}$ made by us not fully accidentally in 1994, the two-loop electroweak corrections to $K \rightarrow \pi\nu\bar{\nu}$ can be safely neglected and similarly for other decays considered in this section.

Much harder calculation has been done by Paolo Gambino, Axel Kwiatkowski and Nicolas Pott [58]. They calculated full two-loop electroweak contributions to $B^0 - \bar{B}^0$ mixing, finding also in this case a very small effect in the \overline{MS} scheme.

Coming back to my paper with Gerhard Buchalla on two-loop electroweak corrections to rare K and B decays [56], we warned the readers that our large m_t limit calculation of these corrections could miss the true value of these corrections by a factor of two. Very recently the younger generation of the Munich club, Joachim Brod, Martin Gorbahn and the youngest member of our club Emanuel Stamou, performed full two loop electroweak calculation to $K \rightarrow \pi\nu\bar{\nu}$ decays, basically reaching the conclusion of our large m_t limit calculation but reducing further the theoretical uncertainty [57]. Last but not least the electroweak contributions to the charm part in $K^+ \rightarrow \pi^+\nu\bar{\nu}$ has been calculated by the duo Brod and Gorbahn [166] already in 2008.

Finally, I would like to note that the NLO and the NNLO summits discussed in this section similarly to the $\Delta F = 2$ summits within the SM have only been reached by the members of the MNLC. However, in Sections 6 and 8 we will discuss summits which have also been conquered by other groups in particular those led by Christoph Greub.

6 The $B \rightarrow X_s\gamma$ Decay: The K2 of Weak Decays

6.1 Preliminaries

The calculations of NLO and recently NNLO QCD corrections to $B \rightarrow X_s\gamma$ decay are probably the best known to the physics community among all QCD calculations in the field of weak decays. One of the reasons is the fact that the $b \rightarrow s\gamma$ transition was the first penguin mediated transition in B physics to be discovered in 1993 in the exclusive decay channel $B \rightarrow K^*\gamma$ by the CLEO experiment. The inclusive branching ratio has been measured in 1994 by the same group. The other reason is the particular structure of the QCD corrections to this decay that requires a two-loop calculation in order to obtain the anomalous dimension matrix in the LO approximation. Because of this it took six years after the first QCD calculations in ordinary perturbation theory to obtain the correct result for the QCD corrections to $B \rightarrow X_s\gamma$ in the renormalization group improved perturbation theory at LO. It involved 5 groups and 16 physicists. It is not then surprising that the corresponding NLO calculations took nine years. They were dominated by the group around Christoph Greub and by the members of the MNLC although a few other physicists also contributed to this enterprise as I will report below. I will concentrate here on

the inclusive decays as they are theoretically cleaner than the exclusive ones but the effective Hamiltonian is of course common to inclusive and exclusive rates. A nice review of $B \rightarrow X_s \gamma$ including exclusive decays $B \rightarrow K^* \gamma$ and $B \rightarrow \rho \gamma$ can be found in the Flavour Bible [167] and in the very recent review [168]. Some comments on the exclusive radiative decays will be made at the end of this Section and in Section 9.

This effective Hamiltonian for $b \rightarrow s \gamma$ is given at the scale $\mu_b = \mathcal{O}(m_b)$ as follows

$$\mathcal{H}_{\text{eff}}(b \rightarrow s \gamma) = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^6 C_i(\mu_b) Q_i + C_{7\gamma}(\mu_b) Q_{7\gamma} + C_{8G}(\mu_b) Q_{8G} \right], \quad (6.1)$$

where in view of $|V_{us}^* V_{ub}/V_{ts}^* V_{tb}| < 0.02$ we have neglected the term proportional to $V_{us}^* V_{ub}$. Here $Q_1 \dots Q_6$ are the usual four-fermion operators whose explicit form is given in (2.5)–(2.7). The remaining two operators, characteristic for this decay, are the *magnetic-penguins* defined in (2.10).

6.2 LO Efforts

In 1987 two groups [169, 170] calculated $\mathcal{O}(\alpha_s)$ QCD corrections to the $B \rightarrow X_s \gamma$ rate finding a huge enhancement of this rate relative to the partonic result without QCD corrections. In 1987, when $m_t \leq M_W$ was still considered, this enhancement was almost by an order of magnitude but with the increased value of m_t in the 1990's also the partonic rate increased and the dominant additive QCD corrections, although still very important, amount in 2011 roughly to a factor of 2.5.

The additive QCD corrections in question originate in the mixing of the operator Q_2 with the magnetic photon penguin operator $Q_{7\gamma}$ that is directly responsible for the decay $b \rightarrow s \gamma$. The calculation of the relevant anomalous dimensions at LO is a two-loop affair and consequently it took some time before the correct result has been obtained. In 1988 Grinstein et al. [171] and the Canadian group [172, 173] calculated the renormalization group improved QCD corrections at LO to $B \rightarrow X_s \gamma$ using the NDR and the DRED scheme, respectively. The results disagreed with each other. This was clearly a surprise as the LO result for the Wilson coefficients cannot depend on the renormalization scheme.

In 1990, Cella et al. [174] confirmed the NDR result of Grinstein et al., and extended it to include the mixing of Q_2 with Q_{8G} . The fourth calculation was done by a rising Polish star, Mikolaj Misiak [176, 177], who in a solo climb evaluated LO QCD corrections to $B \rightarrow X_s \gamma$ decay and NLO corrections to $B \rightarrow X_s l^+ l^-$ decay. I will discuss the last calculation in the next section. In these papers Mikolaj found the fourth LO result for the decay in question and explained why the previous NDR calculations were incomplete. The fifth calculation was done by Adel and Yao [187] who ignored the evanescent operator contributions [177], and for this sole reason failed to reach the correct final result. Other papers of this period contributing to this discussion and formulating effective Hamiltonian for $b \rightarrow s \gamma$ transitions are [178, 179].

Mikolaj has solved all the qualitative issues but being alone he did not succeed to reach this LO summit. This was done in the summer of 1993 by the Rome group led by Guido Martinelli with the participation of a rising Italian star, Luca Silvestrini, whose PhD thesis amounted precisely to solving this problem [180, 181]. This result has been soon confirmed by the Pisa group around Curci [182, 183], and subsequently by Mikolaj in an erratum to his second paper [177].

Each of the LO calculations preceeding the final one had its own missed subtleties. One of the tricky points was that while in the HV scheme the one-loop matrix elements of the QCD penguin operators $M_i = \langle s\gamma | Q_i | b \rangle$ with $i = 3 - 6$ vanish including finite parts, this does not happen in the NDR scheme in which the finite pieces of M_5 and M_6 are different from zero. Combining this one-loop information with the genuine two-loop calculations of the non-diagonal elements of the anomalous dimension matrix involving Q_i with $i = 1 - 6$ and $Q_{7\gamma}$ and Q_{8g} allowed to show that scheme independent leading order result for the Wilson coefficients of the operators $Q_{7\gamma}$ and Q_{8g} could be obtained. This important solution prompted Mikolaj Misiak to extend this analysis to the DRED scheme [184] and him, Stefan Pokorski, Manfred Münz and myself [185] to introduce effective anomalous dimension matrices that were automatically renormalization scheme independent in the leading order even in the $B \rightarrow X_s \gamma$ decay.

Thus by the summer of 1993 the correct results for the Wilson Coefficients relevant for $B \rightarrow X_s \gamma$ and $B \rightarrow X_s g$ decays have been known at LO. However, in this year an important observation was made by Ahmed Ali, Christoph Greub and Thomas Mannel [186]: the LO rate for $B \rightarrow X_s \gamma$ exhibited a very large renormalization scale dependence. Changing the scale μ_b in the Wilson coefficient from $m_b/2$ to $2m_b$ changed the rate of $B \rightarrow X_s \gamma$ by roughly 60% making a detailed comparison of theory with experiment impossible. In 1993 this was not yet a problem as the inclusive rate was unknown experimentally at that time but the discovery of the decay $B \rightarrow K^* \gamma$ by the CLEO collaboration in the summer of 1993 was a signal that an inclusive rate will be known soon as well. In any case this problem applies to $B \rightarrow K^* \gamma$ as well.

The large μ_b dependence found at LO in this decay is actually not surprising. Afterall the QCD effects in this decay are very large which can be traced back, at least in part, to the large anomalous dimensions of the dipole operators.

In the summer of 1993 motivated by the work of Ali, Greub and Mannel I started to look at the steps necessary to do a complete NLO analysis of $B \rightarrow X_s \gamma$ decay with the aim to reduce the strong μ_b dependence found by them at LO. Manfred Münz, my very good PhD student, joined me in this enterprise but we were not sure that just making an outline of this particular NLO calculation would be sufficient for a publication. Fortunately Mikolaj Misiak, who joined my group in 1993, and Stefan Pokorski, who was a visitor at the MPI for Physics at that time, had a complementary problem. They were investigating parametric uncertainties (α_s, m_b, m_c) in the $B \rightarrow X_s \gamma$ decay but similarly to us were not sure that such an analysis would be sufficient for a publication. Once we discovered our “problems”, it was clear that joining our efforts could

result in a useful paper. This turned out to be an excellent decision. Our paper [185] appeared in November 1993, just seven months before the CLEO's discovery of $B \rightarrow X_s \gamma$ rate and became the standard reference in subsequent papers in which the actual climbs of the NLO $B \rightarrow X_s \gamma$ summit have been done.

6.3 NLO Efforts

6.3.1 The Basic Structure

The complete NLO calculation for $B \rightarrow X_s \gamma$ decay consists of three rather difficult steps:

Step 1

The calculation of $\mathcal{O}(\alpha_s)$ corrections to Wilson coefficients of $Q_{7\gamma}$ and Q_{8g} operators at $\mu = \mathcal{O}(M_W)$. The coefficients for Q_i ($i = 1, \dots, 6$) at this order are known from $\Delta F = 1$ non-leptonic Hamiltonian discussed in Section 3.

Step 2

The calculation of the relevant 8×8 anomalous dimension matrix at $\mathcal{O}(\alpha_s^2)$. The 6×6 sub-matrix involving the operators Q_i ($i = 1, \dots, 6$) are known from $\Delta F = 1$ non-leptonic Hamiltonian discussed in Section 3.

Step 3

Calculation of the matrix elements $\langle s\gamma | Q_i | b \rangle$ with $i = 1, \dots, 8$ in perturbation theory in α_s .

In what follows I will discuss these three steps one by one in the order as given above.

6.3.2 Wilson Coefficients of $Q_{7\gamma}$ and Q_{8g} at $\mu = \mathcal{O}(M_W)$

The calculation of these Wilson coefficients is much harder than the calculations of Wilson coefficients discussed in Section 5 due to the presence of external photons and gluons: the external momenta cannot be set to zero. The two-loop diagrams are then calculated with non-vanishing external momenta of quarks, the photon and the gluon, the result is expanded to second order in external momenta and masses and is matched to the result of the corresponding effective theory calculation. The first calculation of $C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ at $\mathcal{O}(\alpha_s)$ has been performed by Adel and Yao already in 1993 [188]. Unfortunately it was done in the Zimmermann's renormalization scheme that was rather unfamiliar to phenomenologists and consequently in 1993 this result was not noticed by many.

Right at the beginning of 1997, Axel Kwiatkowski (one of my assistants), my PhD student Nicolas Pott and myself decided to calculate these coefficients in the NDR scheme, that by then became the standard scheme for all NLO calculations. However, in March 1997 a paper by Christoph Greub and Thobias Hurth [190] appeared on the Los Alamos server, in which they calculated $\mathcal{O}(\alpha_s)$ corrections to $C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ in the NDR scheme and moreover demonstrated that their result was compatible with the one of Adel and Yao. As by that time we invested already two and a half months in this calculation, this was truly an unpleasant surprise.

Fortunately it turned out that we still could contribute something to this calculation. Greub and Hurth regulated the infrared divergences in the full and effective theory by using dimensional regularization but to our surprise performed the matching in four dimensions. As a result of this, infrared divergences did not cancel in the process of matching and they had to argue that they will be removed by including bremsstrahlung corrections. However, they did not demonstrate this but only showed that if this left-over divergence was removed by hand, the resulting finite result in the NDR scheme was compatible with the Adel and Yao result. I am mentioning this here to emphasize that the inclusion of gluon bremsstrahlung effects is really unnecessary to obtain the correct result because on general grounds the calculation of Wilson coefficients can be done by choosing any external states. For this reason we continued our calculation and demonstrated in detail that performing the full calculation in $4 - 2\epsilon$ dimensions, including the matching, directly led to the final result of Greub and Hurth without any handwaving arguments for the disappearance of infrared divergences. These cancellations are very clearly seen in the expressions presented in [191, 192]. In any case by the summer of 1997 three groups found the Wilson coefficients $C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ and consequently the first step of this K2 climb was completed. Few months later an Italian group consisting of Marco Ciuchini, Giuseppe Degrandi, Paolo Gambino and Gian Giudice [67] confirmed these result, working in the off-shell operator basis in contrast to the on-shell basis used by us and Greub and Hurth.

Before continuing I would like to emphasize that in spite of my critical remarks on Greub-Hurth calculation in question both authors played very important roles in the study of QCD corrections to $B \rightarrow X_s \gamma$ and $B \rightarrow X_s l^+ l^-$ decays. In particular Christoph Greub is one of the great masters of these decays and his group made important contributions here both in the SM and beyond it. In this context his numerous analyses of these decays within 2HDM and the MSSM with Francesca Borzumati, the leading lady of NLO QCD calculations in weak decays, should be mentioned [65, 75, 66].

6.3.3 Anomalous Dimension Matrix

The anomalous dimension matrix relevant for the $B \rightarrow X_s \gamma$ decay at the NLO level consists of the 6×6 two-loop mixing matrix of four-fermion operators (Q_1, \dots, Q_6) discussed already in Section 3, the two-loop 2×2 matrix describing the evolution and mixing under renormalization of magnetic operators $Q_{7\gamma}$ and Q_{8g} and finally the three-loop 6×2 matrix describing the mixing between (Q_1, \dots, Q_6) and ($Q_{7\gamma}, Q_{8g}$).

In 1994 the two-loop 2×2 and three-loop 6×2 matrices were still unknown. Mikolaj Misiak and my PhD student Manfred Münz decided to perform the first of these calculations in early 1994. While certainly not an easy task, it has been achieved by developing a new technique of regularizing IR divergences with a common spurious mass parameter. Such a regularization for gluon lines had been previously thought to be prohibited because it breaks the QCD gauge invariance. However, the breaking turns out to be harmless for the RGE parameter calculations

in mass-independent regularization schemes, so long as subdivergences are treated in a careful manner [196]. Understanding this fact was a milestone for subsequent calculations of beta functions and anomalous dimensions at the three- and four-loop levels. The paper of Mikolaj and Manfred appeared on the Los Alamos server in September 1994 [193]. Two months later, Mikolaj presented the results at the Ringberg meeting, triggering objections concerning the IR regularization from David Broadhurst and Kostja Chetyrkin – the great masters in difficult multi-loop integrals. However, Kostja realized very quickly that the method is correct, and offered his help in continuation of the NLO project at the three-loop level.

The three-loop calculation of the 6×2 matrix in question can certainly be regarded as the most spectacular achievement in the field of higher order QCD calculations in weak decays in the 1990's. Kostja provided soon a very efficient recursive formula for three-loop integrals that allowed to begin this project. With 800 three-loop Feynman diagrams this was still a very complicated project led by Mikolaj with a great help from Manfred. Additional difficulty was the treatment of γ_5 at the three-loop level. The calculation in the t'Hooft-Veltman scheme would be simply too much time-consuming and Mikolaj decided to use the NDR scheme. Unfortunately my technique (see Section 3) to avoid the dangerous traces with γ_5 , so successful at the two-loop level, fails at three loops. Kostja insisted that it must be possible to define the effective theory in such a manner that no traces with γ_5 occur, as in the corresponding SM diagrams. Following this advice, Mikolaj replaced the standard basis of four-fermion operators (Q_1, \dots, Q_6) by another (rather complicated looking) set of operators, that allowed him, Manfred and Kostja to complete this project without any γ_5 problems. The result appeared first in [197] and the details of the calculation have been published in [196, 99].

At this point I should mention that the calculation in question, was the first three-loop calculation in the field of weak decays. It's complexity required the use of powerful PC's and has been fully done by using computer programs for algebraic manipulations like [110]. Such programs have been subsequently further developed by my PhD students, Ulrich Haisch, Christoph Bobeth, Martin Gorbahn, and Thorsten Ewerth and by my assistant Jörg Urban, so that similar techniques could be used during the last decade for the calculation of two-loop contributions in supersymmetric theories. More about it later.

The three-loop calculation described above together with the initial conditions discussed in 6.3.2 and the two-loop matrix elements of the relevant current-current operators discussed below allowed in 1996 for the first time an almost complete (see below) NLO analysis of the $B \rightarrow X_s \gamma$ decay [197]. Since then all analyses of this decay used the results of [197]. It was then fortunate that this result has been confirmed by Paolo Gambino, and two of my PhD students Martin Gorbahn and Ulrich Haisch in 2003 [194], who subsequently extended this very tedious calculations to other operators as discussed in section 7 below. They confirmed also the 1994 results of Mikolaj and Manfred for two-loop mixing of the dipole operators.

Table 5: $B \rightarrow X_s \gamma$ at NLO and NNLO. $\hat{\gamma}(\text{Mixing})$ stands for the mixing between 4-quark operators and magnetic penguins. For more references to $B \rightarrow K^*(\rho) \gamma$ see text.

Decay	NLO	NNLO
$C_i(M_W)$	[188, 190, 191, 192, 67]	[54, 55, 198]
$\hat{\gamma}(Q_{7\gamma}, Q_{8G})$	[193, 194]	[199]
$\hat{\gamma}(\text{Mixing})$	[197, 196, 194]	[195]
Matrix Elements	[200, 201, 202, 203, 204, 205]	[210]-[224]
$B \rightarrow K^*(\rho) \gamma$	[241, 225, 306]	[226, 227]

6.3.4 Matrix Elements

The final step of the NLO program for the $B \rightarrow X_s \gamma$ decay is the calculation of the matrix elements $\langle s \gamma | Q_i | B \rangle$. The bremsstrahlung contributions to the matrix elements of $Q_{7\gamma}$, Q_{8g} and Q_2 have been calculated by Ali and Greub in 1995 [200]. This calculation has been confirmed by my diploma student Nicolas Pott, who extended Ali and Greub calculation to penguin operators [201]. In these papers also one-loop virtual contributions of the matrix elements of $Q_{7\gamma}$ and Q_{8g} have been calculated.

Much harder is the calculation of the two-loop matrix elements of the four-quark operators, that are relevant for the NLO analysis of $B \rightarrow X_s \gamma$. In particular, as already stressed in our 1993 paper [185], the μ -dependence of two-loop matrix elements should significantly cancel the strong μ_b -dependence of the LO branching ratio, pointed out by Ali, Greub and Mannel in 1993 [186].

The difficulty in this calculation is that an expansion in external momenta cannot be made and one has to face a two-loop calculation with full kinematics involved. On the other hand in the case of the matrix element of the current-current operator Q_2 an expansion in powers of m_c^2/m_b^2 can be made. The first calculation of this type, using the technique of Gegenbauer polynomials, has been done by Christoph Greub, Tobias Hurth and Daniel Wyler already in 1996 [202, 203]. As anticipated in 1993, this contribution decreased the strong μ_b -dependence of the rate from $\pm 30\%$ found by Ali, Greub and Mannel down to approximately $\pm 5\%$.

In the summer of 1997, while lecturing at the Les Houches summer school on the weak effective Hamiltonian and higher order QCD corrections, I started thinking about repeating the 1996 calculation of Greub, Hurth and Wyler. As many of the members of the MNLC were involved already in other projects, I started looking for additional collaborators. Fortunately among the many very good students of this summer school, there was one who was already a two-loop climber at that time: Andrzej Czarnecki. Already in 1997 Andrzej had on his account two very important two-loop calculations [132] relevant for the extraction of the CKM element V_{cb} and knew another technique that could in principle be used for the calculation of the matrix

elements in question: the technique of heavy mass expansions. After a short discussion we agreed to join our forces as soon as we complete the projects we were both involved at that time. Accidentaly our discussion has been documented in the form of a photo taken by Rajan Gupta, that can be found in the proceedings of the school just before my contribution.

However, in the next two years we were both very busy with other projects. Among other things Andrzej with his well known $g - 2$ calculations and I with writing up my Les Houches lectures [16] which thanks to great generosity of Rajan Gupta amounted finally to 250 pages.

In 2000 Andrzej Czarnecki and me met again and decided to increase our team. We were joined by the master Mikolaj Misiak and by experienced, already at that time, young NLO climber Jörg Urban. In the first climb [204] led by Andrzej Czarnecki we confirmed the result of Greub, Hurth and Wyler by using the method of heavy mass expansions, generalizing their result to a few additional terms in the expansion in m_c^2/m_b^2 that was found to converge rapidly. In the second climb [205] led by Jörg and Mikolaj we succeeded to express all the two-loop matrix elements in terms of four compact two- and three-fold Feynman-parameter integrals. It allowed us to complete the NLO $B \rightarrow X_s \gamma$ project by calculating its last ingredient: the two-loop matrix elements of QCD penguin operators. This last part involved few diagrams with internal b-quark, implying the replacement of m_c^2/m_b^2 by 1 and making the expansions used in [202, 203] and [204] useless for the calculation of the matrix elements in question.

This last paper [205] has been completed in the spring of 2002, just a few months before the celebration of Stefan Pokorski's 60th birthday at the Architects house in Kazimierz in Poland. Thus eight and a half years after the outline of the NLO $B \rightarrow X_s \gamma$ program [185], Mikolaj and me could summarize the result of these efforts in the volume of Acta Physica Polonica dedicated to Stefan's 60th birthday [206].

Before entering the NNLO story of this decay let me mention that very recently tree-level contribution $b \rightarrow u \bar{u} s \gamma$ to $B \rightarrow X_s \gamma$ has been calculated by Kamiński, Misiak and Poradziński [207]. However, for the photon energy cutoff of $E_0 = 1.6$ GeV or higher this contribution amounts to 0.4% at the level of the branching ratio.

6.4 $B \rightarrow X_s \gamma$ at NNLO

The story of higher-order calculations of the $B \rightarrow X_s \gamma$ rate is not finished yet. While these calculations decreased the μ_b -dependence present in the LO expressions significantly, a new uncertainty has been pointed out by Paolo Gambino and Mikolaj Misiak in 2001 [208]. It turns out that the $B \rightarrow X_s \gamma$ rate suffers at the NLO from a significant, $\pm 6\%$, uncertainty due to the choice of the charm quark mass in the two-loop matrix elements of the four quark operators, in particular in $\langle s \gamma | Q_2 | B \rangle$.

In the first calculations [202, 203, 204, 205] the pole charm quark mass has been used. But as stressed by Mikolaj and Paolo there is no particular reason why the pole mass should be used instead of the \overline{MS} mass, and in fact they argued that the latter choice is more appropriate in

the case at hand as charm appears only as internal particle.

While the arguments of Mikolaj and Paolo are very plausible, it is clear that finally the $B \rightarrow X_s \gamma$ rate cannot depend on the choice of the charm quark mass, even if this dependence is significant at the NLO level.

Thus in order to remove or reduce this uncertainty a NNLO calculation is necessary, a truly formidable task. It requires various calculations in three steps:

Step 1:

$C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ at the three-loop level and those of $C_i(M_W)$ ($i = 1 - 6$) at two-loop level,

Step 2:

Three-loop 6×6 and 2×2 anomalous dimension matrices of the operators (Q_1, \dots, Q_6) and $(Q_{7\gamma}, Q_{8g})$ as well as the mixing between these two sets of operators at the four-loop level!

Step 3:

The matrix elements $\langle s\gamma | Q_i | B \rangle$ ($i = 1, \dots, 6$) at the three loop level and of the corresponding matrix elements of the magnetic (dipole) operators $(Q_{7\gamma}, Q_{8g})$ at the two-loop level.

In the spring of 2003, another workshop in the Ringberg castle took place. This workshop, organized by Andre Hoang, Gerhard Buchalla, Thomas Mannel and myself, gathered experts in heavy flavour physics and had a much broader spectrum of topics than the seminal 1988 workshop at which the NLO story has been initiated. At this workshop Uli Haisch presented the first steps of the three-loop calculations of the anomalous dimensions of Q_i ($i = 1, \dots, 6$) relevant for $B \rightarrow X_s \gamma$ at the NNLO level, and of three-loop mixing between Q_i ($i = 1, \dots, 6$) and the semi-leptonic operators relevant for the decay $B \rightarrow X_s l^+ l^-$ at NNLO. We will discuss this last topic in Section 8. Moreover, Mikolaj Misiak outlined three-loop calculations of the relevant matrix elements and of the Wilson coefficients $C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ [209].

In the following years after this 2003 workshop, considerable progress in the NNLO program of $B \rightarrow X_s \gamma$ has been made, and the $B \rightarrow X_s \gamma$ rate at NNLO can already be estimated. It was an effort of more than 17 theorists [228] and required a number of calculations over the period of six years by several groups. As I was not involved in this impressive project I will leave the description to the participants of this K2-like climb. I found in particular the summaries of Mikolaj Misiak [229], who led these efforts and of Uli Haisch [230] very informative and clear. I would like to thank Mikolaj for improving my insight in these calculations.

Let me then only pay the tribute to the successful climbers of the K2 of weak decays by listing hopefully most important calculations which led the team of Mikolaj Misiak to this important victory. Here the summaries of Uli Haisch and Mikolaj Misiak quoted above were very helpful. Explicitly:

- $C_{7\gamma}(M_W)$ and $C_{8g}(M_W)$ at the three-loop level have been calculated by Misiak and Steinhäuser [198] and those of $C_i(M_W)$ ($i = 1 - 6$) at two-loop level by Bobeth, Misiak and Urban [54].

- The three-loop 2×2 anomalous dimension matrix of dipole operators has been calculated by Gorbahn, Haisch and Misiak [199] and the three loop 6×6 anomalous dimension matrix of $(Q_1, ..Q_6)$ operators by Gorbahn and Haisch [93]. Finally the four loop mixing of (Q_1, Q_2) operators with the dipole operators has been found by Czakon, Haisch and Misiak [195]. The latter one is the most impressive part of this grand NNLO project. It has involved the computation of more than 20000 four-loop diagrams and required a mere computing time of several months on around 100 CPU's.
- A very difficult part of NNLO calculation turned out to be related to the last step of the program, the calculation of the matrix elements to the desired order. This calculation is not yet completed but a fantastic progress has been made as recently summarized very systematically by Mikolaj Misiak [229].

As several groups took part in the latter step it is essential to refer to all the existing calculations. To this end I found it most convenient to shorten the most recent summary of Misiak [229]. The discussion of non-perturbative contributions is also included there.

Once the Wilson coefficients have been calculated to the desired ordered, following Misiak the partonic decay rate is evaluated according to the formula

$$\Gamma(b \rightarrow X_s \gamma)_{E_\gamma > E_0} = N \sum_{i,j=1}^8 C_i(\mu_b) C_j(\mu_b) G_{ij}(E_0, \mu_b), \quad (6.2)$$

where $N = |V_{ts}^* V_{tb}|^2 (G_F^2 m_b^5 \alpha_{\text{em}}) / (32\pi^4)$. At the Leading Order (LO), we have $G_{ij} = \delta_{i7} \delta_{j7}$, while the $\mathcal{O}(\alpha_s)$ NLO contributions have been summarized above and in [206].

At the NNLO, it is sufficient to restrict the attention to $i, j \in \{1, 2, 7, 8\}$ because the Wilson coefficients of QCD penguin operators $C_{3,4,5,6}(\mu_b)$ are very small. Treating the two similar operators Q_1 and Q_2 as a single one (represented by Q_2) we list the papers where six independent cases of the NNLO contributions to G_{ij} have been calculated.

First G_{77} , G_{78} and G_{27} involve the photonic dipole operator $Q_{7\gamma}$. While G_{77} was found already several years ago [211, 212, 213, 214, 215], the complete calculation of G_{78} has been finalized only very recently [216, 217]. Evaluation of G_{27} is still in progress (see below).

The remaining three cases (G_{22} , G_{28} and G_{88}) receive contributions from different classes of diagrams. Diagrams involving two-body final states are IR-convergent and are just products of the known NLO amplitudes. Three- and four-body final state contributions remain unknown at the NNLO beyond the BLM approximation [231]. The BLM calculation for them has been completed very recently [219, 220] providing new results for G_{28} and G_{88} , and confirming the old ones [218] for G_{22} . The overall NLO + (BLM-NNLO) contribution to the decay rate from three- and four-body final states in G_{22} , G_{28} and G_{88} remains below 4% due to the phase-space suppression by the relatively high photon energy cut E_0 . Thus, the unknown non-BLM effects here can hardly cause uncertainties that could be comparable to higher-order $\mathcal{O}(\alpha_s^3)$

uncertainties in the dominant terms (G_{77} and G_{27}). Misiak concludes that the considered G_{ij} are known sufficiently well.

Thus the only contribution that is numerically relevant but yet unknown at the NNLO is G_{27} . So far, it has been evaluated for arbitrary m_c in the BLM approximation [221, 218] supplemented by quark mass effects in loops on the gluon lines [224]. Non-BLM terms have been calculated only in the $m_c \gg m_b/2$ limit [222, 223], and then interpolated downwards in m_c using BLM-based assumptions at $m_c = 0$. Such a procedure introduces a non-negligible additional uncertainty to the calculation, which Misiak estimates to be at the level $\pm 3\%$ in the decay rate.

As a first attempt to improve the situation, a calculation of G_{27} at $m_c = 0$ has been undertaken [232, 233, 234]. Moreover a recently started calculation [235] for arbitrary m_c is supposed to cross-check the $m_c = 0$ result and, at the same time, make it redundant, because no interpolation in m_c will be necessary any more. Good luck!

The references to $B \rightarrow X_s \gamma$ NLO and NNLO calculations are collected in Table 5. In several of these papers also the less important decay $B \rightarrow X_s \text{gluon}$ has been analyzed. We refer in particular to [181, 188, 190, 55, 236, 237] and more recent reviews [167, 168]. Also NNLO calculation for $B \rightarrow X_s \gamma \gamma$ has been performed in [323].

These are truly impressive calculations and achievements. While I did not participate in this NNLO calculation I am very satisfied that 8 members of the Munich club and my collaborators took part in this grand project. These are Mikolaj Misiak, Paolo Gambino, Andrzej Czarnecki, Jörg Urban and four of my former PhD students: Christoph Bobeth, Martin Gorbahn, Ulrich Haisch and Thorsten Ewerth.

6.5 $B \rightarrow X_d \gamma$

The perturbative QCD corrections to the inclusive decay $B \rightarrow X_d \gamma$ have been analyzed in [238, 239, 240] and their structure is totally analogous to the case of the $b \rightarrow s \gamma$ transition up to obvious changes in flavour indices. But as $\lambda_u = V_{ub}V_{ud}^*$ for $b \rightarrow d \gamma$ is not small with respect to $\lambda_t = V_{tb}V_{td}^*$ and $\lambda_c = V_{cb}V_{cd}^*$, one also has to take into account the terms proportional to λ_u .

6.6 Exclusive Radiative Decays

The effective Hamiltonians for inclusive radiative decays apply of course to exclusive decays as well. Here the additional complications are hadronic uncertainties. As these are beyond the scope of this writing I just refer to selected papers [225, 226, 241, 242, 243, 244, 245] and recent reviews [167, 168].

7 $K_L \rightarrow \pi^0 l^+ l^-$ at NLO

7.1 Effective Hamiltonian

In this and the following section we will discuss two well known decays: $K_L \rightarrow \pi^0 l^+ l^-$ and $B \rightarrow X_s l^+ l^-$. The reason for collecting these decays close to each other is related to the fact that the results for the first one are very helpful in reducing the work necessary to obtain the QCD corrections to the second decay.

Thus the effective Hamiltonian for $K_L \rightarrow \pi^0 e^+ e^-$ given in (7.1) includes in addition to the operators $Q_1 \dots Q_6$ the semi-leptonic operators Q_{7V} and Q_{7A} defined in (7.2). On the other hand, the effective Hamiltonian for $B \rightarrow X_s \ell^+ \ell^-$ given in (8.1) can be considered as the generalization of the effective Hamiltonian for $B \rightarrow X_s \gamma$ in (6.1) to include the semi-leptonic operators Q_{9V} and Q_{10A} defined in (8.2). Evidently the operators in (7.2) and (8.2) behave identically under QCD interactions that are flavour blind. Only the renormalization scales differ.

The effective Hamiltonian for $K_L \rightarrow \pi^0 e^+ e^-$ at scales $\mu < m_c$ is then given as follows:

$$\mathcal{H}_{\text{eff}}(K_L \rightarrow \pi^0 e^+ e^-) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left[\sum_{i=1}^{6,7V} [z_i(\mu) + \tau y_i(\mu)] Q_i + \tau y_{7A}(M_W) Q_{7A} \right], \quad (7.1)$$

where the four-quark operators Q_1, \dots, Q_6 are familiar by now and the new operators Q_{7V} and Q_{7A} are given by

$$Q_{7V} = (\bar{s}d)_{V-A}(\bar{e}e)_V, \quad Q_{7A} = (\bar{s}d)_{V-A}(\bar{e}e)_A. \quad (7.2)$$

There are three contributions to this decay: CP conserving, directly CP-violating and indirectly CP-violating. In 1993 when Markus Lautenbacher, Mikolaj Misiak, Manfred Münz and myself started looking at this decay it was not clear which of these components was the dominant one although there was some hope that the theoretically cleanest component, the directly CP-violating one, was the dominant one. The hope was partially based on the fact that at lowest order in electroweak interactions (single photon, single Z-boson or double W-boson exchange), this decay takes place only if the CP symmetry is violated [246]. The CP conserving contribution to the amplitude comes from a two photon exchange which, although higher order in α , could in principle be sizable.

Extensive work over the last 15 years on the non-perturbative CP conserving component and indirectly CP-violating one shows that within the SM the latter component dominates followed by the interference between the two CP-violating components. The non-perturbative part calculated by means of chiral perturbation method turns out to be smaller than these components. Most recent discussions of this decay including also $K_L \rightarrow \pi^0 \mu^+ \mu^-$ can be found in [247, 248, 249, 250].

7.2 The Structure of NLO Corrections

Back to 1993. Our goal was to calculate the coefficients y_{7V} and y_{7A} that are relevant for the directly CP-violating component at the NLO level. In order to see what was known at that time let us introduce \tilde{y}_i through

$$y_i = \frac{\alpha}{2\pi} \tilde{y}_i \quad (7.3)$$

with \tilde{y}_i having the following structure according to PBE

$$\tilde{y}_{7V} = P_0 + \frac{Y_0(x_t)}{\sin^2 \theta_W} - 4Z_0(x_t), \quad (7.4)$$

$$\tilde{y}_{7A} = -\frac{1}{\sin^2 \theta_W} Y_0(x_t) \quad (7.5)$$

with Y_0 and Z_0 being two one-loop master functions in the SM.

The next-to-leading QCD corrections to the coefficients above enter only P_0 . The top dependent terms were known from the work of Fred Gilman and collaborators although they have written them in a different manner as Fred was involved in this field before PBE was introduced. Moreover P_0 at LO was known as well.

The structure of QCD corrections to this decay belongs to class 7 introduced in Section 2 which is a bit special. In this language what was known in 1993 was the leading order represented by an $1/\alpha_s$ in P_0 and the top mass dependent part of the NLO represented by Y_0 and Z_0 .

Our goal was to calculate the remaining, top mass independent, NLO corrections to P_0 that are of $\mathcal{O}(1)$ in the α_s expansion. The most difficult part in this calculation is the evaluation of the two-loop anomalous dimension matrix involving the operators $Q_1 \dots Q_6$ and the semi-leptonic operator Q_{7V} **. The 6×6 submatrix was already known from the work of Munich and Rome groups and only mixing between these six operators and Q_{7V} had to be calculated at two-loop level: as the later operator has no anomalous dimension only *six* elements of this matrix had to be calculated. The relevant diagrams are given in Fig. 11. After all the hard calculations that we performed before, this one turned out to be a relatively easy one and after a month of work the two-loop 7×7 matrix relevant for this decay and subsequently P_0 at NLO was known.

This calculation reduced a number of ambiguities present in leading order analyses [251, 252] and enhanced P_0 by roughly 30%. The inclusion of NLO QCD effects made also a meaningful use of $\alpha_{\overline{MS}}$ in this decay possible. Our paper appeared in February 1994 [253]. The two-loop mixing of all the four-quark operators with Q_{7V} agreed with an earlier solo calculation of one

** Q_{7A} has no anomalous dimensions and in addition does not mix with the remaining operators. Therefore the only scale uncertainty in its Wilson coefficient originates in $m_t(\mu_t)$ in $Y_0(x_t)$. This uncertainty is practically removed through the inclusion of QCD corrections to this function that we have made in the context of NLO calculation for $K_L \rightarrow \mu^+ \mu^-$ in Section 5. In $K_L \rightarrow \pi^0 e^+ e^-$ these corrections appear first at the NNLO level.

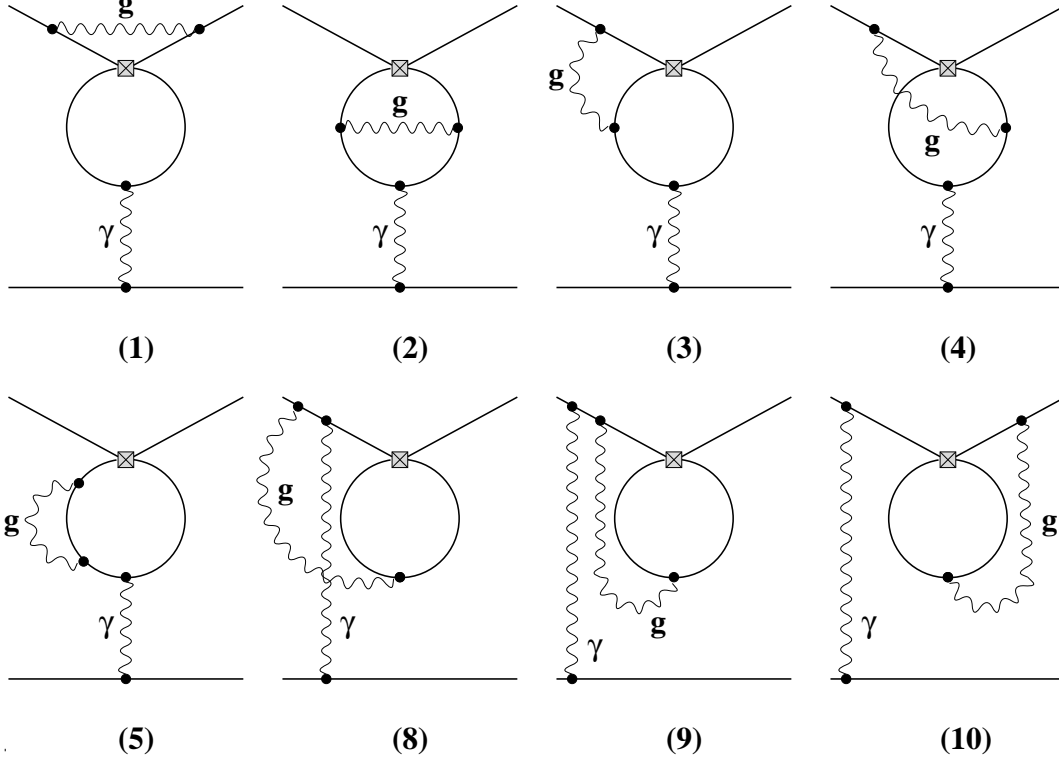


Figure 11: Two-loop penguin diagrams contributing to $\gamma_{i7}^{(1)}$, $i = 1, \dots, 6$. Square vertices stand for two types of penguin insertions. Possible left-right reflected diagrams are not shown. The numbering of the diagrams corresponds to the notation in Figs. 4 and 6.

of us [177], once all the convention differences had been taken into account. Appendix B of our common paper [253] contains a detailed description of this issue. Important information on the NLO analysis of $B \rightarrow X_s l^+ l^-$ in [177] is contained there. I will comment on that in the next section.

8 $B \rightarrow X_s l^+ l^-$ at NLO and NNLO

8.1 Effective Hamiltonian

The rare decays $B \rightarrow X_s l^+ l^-$ have been the subject of many theoretical studies in the framework of the SM and its extensions. Most recent reviews with a very good collection of references can be found in [167, 168].

The relevant effective Hamiltonian at scales $\mu = O(m_b)$ is given by

$$\mathcal{H}_{\text{eff}}(b \rightarrow s \mu^+ \mu^-) = \mathcal{H}_{\text{eff}}(b \rightarrow s \gamma) - \frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} [C_{9V}(\mu) Q_{9V} + C_{10A}(M_W) Q_{10A}] , \quad (8.1)$$

Table 6: $K_L \rightarrow \pi^0 l^+ l^-$ and $B \rightarrow X_s l^+ l^-$ at NLO and NNLO. For $B \rightarrow K^*(\rho) l^+ l^-$ see text.

Decay	NLO	NNLO
$K_L \rightarrow \pi^0 e^+ e^-$	[253]	
$B \rightarrow X_s \ell^+ \ell^-$		
$C_i(M_W)$	[177, 254]	[54]
$\hat{\gamma}(\text{Mixing})$	[177, 254]	[194]
Matrix Elements	[177, 254]	[256, 257, 258, 259, 260, 261] [262, 263, 264, 265, 266, 267]
$B \rightarrow X_d \ell^+ \ell^-$	[177, 254]	[268, 269]
$B \rightarrow K^* \mu^+ \mu^-$	[270, 271, 272]	[241, 226]

where we have neglected the term proportional to $V_{us}^* V_{ub}$ and $\mathcal{H}_{\text{eff}}(b \rightarrow s\gamma)$ is given in (6.1). In addition to the operators relevant for $B \rightarrow X_s \gamma$, there are two new operators:

$$Q_{9V} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_V, \quad Q_{10A} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_A. \quad (8.2)$$

8.2 $B \rightarrow X_s l^+ l^-$ at NLO

As far as the last two operators are concerned the structure of QCD corrections is very similar to the case of $K_L \rightarrow \pi^0 e^+ e^-$ and one can use our calculations for $K_L \rightarrow \pi^0 e^+ e^-$ stopping the RG evolution from M_W already at $\mu = \mathcal{O}(m_b)$. In addition the matrix element of Q_{9V} that also involves the mixing with $Q_1 - Q_6$ operators had to be evaluated at the NLO level. The latter one-loop calculation was possibly the main new achievement in this paper which I did only with Manfred Münz, once we realized during a lunch in TUM-mensa in Garching that this paper could be completed rather fast [254]. At the same time, Mikolaj Misiak followed the instructions from Appendix B of our common paper [253] to remove a convention mismatch in his earlier NLO analysis of $B \rightarrow X_s l^+ l^-$ [177]. It led him to a phenomenological formula for the so-called effective coefficients that parametrize the decay rate in question. The formula was published in an erratum to [177]. We compared our results prior to publication and found perfect agreement.

While the work on $K_L \rightarrow \pi^0 e^+ e^-$ was harder, our results on $B \rightarrow X_s l^+ l^-$ became the standard reference on NLO QCD corrections to these decays, dominantly because the data on $B \rightarrow X_s l^+ l^-$ improved dramatically in the last 15 years, while the decay $K_L \rightarrow \pi^0 e^+ e^-$ has not been observed yet. In fact [254] is one of the most cited papers in the MNLC.

8.3 $B \rightarrow X_s l^+ l^-$ at NNLO

Around the year 2000 various groups calculated NNLO corrections to $B \rightarrow X_s l^+ l^-$ and $B \rightarrow K^* l^+ l^-$ putting in particular emphasis on the reduction of various scale uncertainties present in

our NLO calculations. An important role in these efforts played the forward-backward asymmetry and the point in the invariant dimuon mass s_0 at which this asymmetry vanishes. The calculation of s_0 is theoretically rather clean for both decays, the feature pointed out by Burdman [255] in the context of $B \rightarrow K^* l^+ l^-$. A meaningful discussion of s_0 begins first at the NLO level. From this point of view the NNLO calculations in question amount to NLO corrections to s_0 reducing considerably the scale dependence of the LO result that we could have discussed already in 1994 but somehow did not.

A compact summary of the NNLO calculations has been recently presented by Hurth and Nakao [168]. As I was not involved in these efforts I will not describe them here in detail but in order to be complete in references as much as possible I will very briefly summarize what has been done for the perturbative contributions to the decays in question. This time the review in [168] turned out to be very useful.

First of all let us stress that for the NNLO calculations of $B \rightarrow X_s l^+ l^-$ many parts of the NLO calculations of $B \rightarrow X_s \gamma$ can be used. This is evident when one compares the formal structure of QCD corrections given in (2.48) for Class 6 relevant for $B \rightarrow X_s \gamma$ with the structure of QCD corrections given in (2.49) for Class 7 to which $B \rightarrow X_s l^+ l^-$ belongs. Indeed, whereas $\mathcal{O}(\alpha_s)$ corrections in the latter decay represent NNLO term, they represent the NLO term in the case of $B \rightarrow X_s \gamma$. Therefore below I will only list the calculations that are specific to $B \rightarrow X_s l^+ l^-$ and were not done in the context of $B \rightarrow X_s \gamma$. Again we divide the calculations in three steps as in previous Sections:

Step 1:

In [54] $\mathcal{O}(\alpha_s)$ corrections to the Wilson coefficient of Q_{9V} have been completed. More explicitly such corrections to the penguin function Z were still missing, while those to the function Y were already known from the calculations for $B_s \rightarrow \mu^+ \mu^-$ described before. In this manner the large matching scale uncertainty of 16% at the NLO level has been practically eliminated. Note that the coefficient of Q_{10A} at this order, represented by the function Y was also known as already mentioned in a footnote few pages before.

Step 2:

The $\mathcal{O}(\alpha_s)$ corrections to the term P_0 have been calculated by Gambino, Gorbahn and Haisch [194] who first generalized my two-loop calculation of the mixing between $(Q_1 - Q_6)$ operators with the semi-leptonic operator Q_{9V} , done in collaboration with Lautenbacher, Misiak and Münz ($\hat{\gamma}_{se}^{(1)}$ in (2.37)) to next order by calculating $\hat{\gamma}_{se}^{(2)}$. At this order also three loop mixing in the $(Q_1 - Q_6)$ sector is required. It has been calculated by Gorbahn and Haisch [93] and entered already $B \rightarrow X_s \gamma$ at the NNLO level.

Step 3:

Let me note that until this point the calculations discussed in this section were fully in the domain of MNLC. However, in Step 3 at the NNLO level other groups dominated the NNLO calculations, in particular Christoph Greub and his powerful army. In fact the four-quark

matrix elements including the corresponding bremsstrahlung contributions have been calculated for the low- q^2 region in [256, 257, 258], bremsstrahlung contribution for the forward-backward asymmetry in $B \rightarrow X_s l^+ l^-$ in [259, 260, 261], and the four-quark matrix elements in the high- q^2 region in [258, 262]. The two-loop matrix element of the operator Q_{9V} has been estimated using the corresponding result in the decay mode $B \rightarrow X_u l \nu$ and also Pade approximation methods [263]; this estimate has been further improved in [266]. For QED corrections we refer to [263, 264, 265, 266]. The SCET methods at NNLO have been used in [267].

For the study of the decay $B \rightarrow X_d l^+ l^-$ we refer to [268, 269]. The analysis is very similar but this time CKM suppressed terms have to be kept as in the case of $B \rightarrow X_d \gamma$ decay.

Finally I would like to mention a paper on NNLO corrections to $B \rightarrow X_s l^+ l^-$ in the MSSM. This is the work done in collaboration with my PhD students Christoph Bobeth and Thorsten Ewerth [81]. The main motivation for this work was the reduction of scale uncertainties in s_0 which differs a bit from the one in the SM and in order to feel this difference perturbative uncertainties have to be under control. I should emphasize that in contrast to the rest of my papers described above, my contribution to this paper was minor. Indeed Christoph and Thorsten should be fully credited for this work.

8.4 $B \rightarrow K l^+ l^-$ and $B \rightarrow K^*(\rho) l^+ l^-$

The NLO and NNLO QCD corrections discussed for inclusive decays can of course be also used for corresponding exclusive decays that are easier to measure. Again as in the case of $B \rightarrow K^*(\rho) \gamma$ formfactor uncertainties matter. In particular $B \rightarrow K^* \mu^+ \mu^-$ decay has been investigated by many authors of whom we can cite only a few. Among the older papers let me just mention [270, 271, 272]. In particular in 1999, Ali et al. calculated the dilepton mass spectrum and A_{FB} in the SM and various SUSY scenarios using naïve factorization and QCD sum rules on the light cone [271]. Later it was shown by Beneke et al. [241, 226] that $B \rightarrow K^* \mu^+ \mu^-$ admits a systematic theoretical description using QCD factorization in the heavy quark limit $m_b \rightarrow \infty$. This limit is relevant for small invariant dilepton masses and reduces the number of independent form factors from 7 to 2. Spectator effects, neglected in naïve factorization, also become calculable. In [273], a calculation of $B \rightarrow K^* \mu^+ \mu^-$ using soft-collinear theory (SCET) was presented.

There are other aspects related to formfactors but this topic is outside the scope of this review and I refer only to most recent papers [274, 275, 276, 277], where the formfactor issues are discussed in detail and various symmetries and asymmetries in the SM and various New Physics models have been studied. In these papers a definite progress on the calculation of formfactors has been achieved. A lot of information can also be found in recent reviews [167, 168]. Finally an important paper discussing theoretical aspects of $B \rightarrow K l^+ l^-$ and $B \rightarrow K^* l^+ l^-$ at large q^2 , in particular in connection with hadron-quark duality is a very recent paper by Beylich, Buchalla and Feldmann [278].

The next pages of this review are dedicated to the NLO and NNLO QCD corrections done in the QCD factorization approach to two-body B decays. This means Gerhard Buchalla is entering the scene and I can collect the energy for the final Section.

9 QCD Factorization for Exclusive B Decays (by G. Buchalla)

9.1 Introduction

The formulation of factorization theorems for exclusive hadronic B -meson decays in 1999 made an entire new class of processes accessible to systematic calculations of higher-order corrections in QCD [279, 280]. These processes include B decays into a pair of light mesons, the prototype of which is $B \rightarrow \pi\pi$, but also rare and radiative decays, such as $B \rightarrow K^*\gamma$ or $B \rightarrow K^*l^+l^-$. In the heavy-quark limit, that is up to relative corrections of order Λ_{QCD}/m_b , the problem of computing exclusive hadronic decay amplitudes simplifies considerably. In this limit the decay amplitudes can be written as hard-scattering kernels, which are *process dependent* but perturbatively calculable, multiplied by hadronic quantities such as $B \rightarrow \pi$ form factors, meson decay constants and light-cone wave functions, which are nonperturbative but *process-independent*. The decomposition into calculable hard contributions and universal hadronic quantities is in full analogy with the factorization of short-distance and long-distance terms that is the basis of almost any application of QCD to high-energy processes. Correspondingly the framework is referred to as QCD factorization for exclusive hadronic B decays, or QCD factorization for short. In the present section we review the factorization formula and give an overview of the NLO and NNLO calculations performed for exclusive B decays. We conclude with a brief outlook on highlights of QCD factorization and special applications to precision flavour physics.

9.2 Factorization formula

The matrix element of an operator Q_i in the effective weak Hamiltonian for the decay of a \bar{B} meson into a pair of light mesons M_1M_2 is given by [279, 280]

$$\begin{aligned} \langle M_1M_2|Q_i|\bar{B}\rangle &= F^{B\rightarrow M_1}(m_2^2) \int_0^1 du T_i^I(u) \Phi_{M_2}(u) + (M_1 \leftrightarrow M_2) \\ &+ \int_0^1 d\xi du dv T_i^{II}(\xi, u, v) \Phi_B(\xi) \Phi_{M_1}(v) \Phi_{M_2}(u) \end{aligned} \quad (9.3)$$

up to power corrections of order Λ_{QCD}/m_b . $F^{B\rightarrow M_{1,2}}(m_{2,1}^2)$ are $B \rightarrow M_{1,2}$ form factors, where $m_{1,2}$ denote the light-meson masses, and Φ_M is the light-cone distribution amplitude for the quark-antiquark Fock state of meson M . Here the light-cone distribution amplitudes are understood to include the decay constant f_M of meson M in their normalization. These quantities define the nonperturbative input needed for the computation of the decay amplitudes in QCD

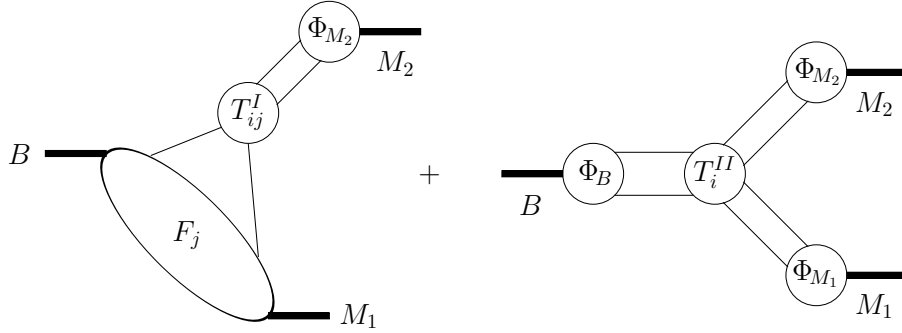


Figure 12: Factorization formula.

factorization. They are simpler than the full matrix element on the l.h.s. of (9.3) and universal in the sense that they appear as well in many other processes, which are different from $\bar{B} \rightarrow M_1 M_2$. The $T_i^I(u)$ and $T_i^{II}(\xi, u, v)$ are the hard-scattering functions, which are process-specific and depend in particular on the operator Q_i . They are calculable by standard methods in perturbative QCD. The formula (9.3) exhibits the factorization of the short-distance kernels T_i and the long-distance hadronic quantities $F^{B \rightarrow M}$ and Φ_M . The factorization of the latter takes, in general, the form of a convolution over the parton momentum fractions $\xi, u, v \in [0, 1]$. A graphical representation of (9.3) is given in Fig. 12, where index j accounts for the possibility of more than a single $B \rightarrow M_1$ form factor. The second term ($\sim T^{II}$) is distinguished from the first ($\sim T^I$) by the participation of the B -meson spectator quark in the hard interaction, indicated by the spectator line entering the kernel T^{II} . The spectator interaction requires the exchange of a hard gluon. T^{II} starts therefore at order α_s , whereas T^I is of order unity, schematically

$$T^I = T_{(0)}^I + \frac{\alpha_s}{4\pi} T_{(1)}^I + \left(\frac{\alpha_s}{4\pi}\right)^2 T_{(2)}^I + \dots, \quad T^{II} = \frac{\alpha_s}{4\pi} T_{(1)}^{II} + \left(\frac{\alpha_s}{4\pi}\right)^2 T_{(2)}^{II} + \dots \quad (9.4)$$

The above description relies on the fact that in the two-body decay of the B meson the final-state particles are necessarily very energetic, with light-like four-momenta, in the rest frame of the B . A meson emitted from the hard interaction, such as M_2 from T^I in Fig. 12, is then described by its light-cone distribution amplitude. At leading power in Λ_{QCD}/m_b the amplitude is determined by the contribution from the light-cone wave function of leading twist, which corresponds to the simplest, two-particle Fock state. Higher Fock states give power-suppressed contributions and are therefore absent in the heavy-quark limit. For example, an additional energetic gluon, collinear to the light-like quark and anti-quark in meson M_2 will generate an additional, far off-shell propagator when attached to the hard process T , which results in a power suppression. The properties of the light-cone wave functions, which vanish at the endpoints ($u = 0, 1$), also imply the suppression of highly asymmetric configurations where one parton carries almost the entire meson momentum and the other parton is soft.

To leading order in QCD, at $\mathcal{O}(\alpha_s^0)$, the factorized matrix element in (9.3) reduces to a particularly simple result. The second term T^{II} is absent at this order and $T^I(u)$ becomes a u -independent constant. Taking the matrix element of operator $Q_1 = (\bar{u}b)_{V-A}(\bar{d}u)_{V-A}$ for $\bar{B} \rightarrow \pi^+\pi^-$ as an example^{††}, the factorization formula then states that

$$\langle \pi^+\pi^- | (\bar{u}b)_{V-A}(\bar{d}u)_{V-A} | \bar{B} \rangle = \langle \pi^+ | (\bar{u}b)_{V-A} | \bar{B} \rangle \langle \pi^- | (\bar{d}u)_{V-A} | 0 \rangle = iF^{B \rightarrow \pi}(0)f_\pi m_B^2 \quad (9.5)$$

This corresponds to the prescription of factorizing the matrix element of the 4-quark operator into a product of matrix elements of bilinear quark currents. Such an ansatz, which has a long history in phenomenological applications [290], thus receives its proper justification in the context of QCD factorization. The approximation in (9.5) means that the emission of the π^- is independent of the remaining $\bar{B} \rightarrow \pi^+$ transition. The intuitive argument for this, namely that the energetic and highly collinear, colour-singlet $\bar{u}d$ pair forming π^- has little interaction with the rest of the process, has been described in [291]. The factorization theorem (9.3) is the formal implementation of this idea and it allows us to compute corrections systematically.

Factorization also works for decays of the type $\bar{B} \rightarrow D^+\pi^-$ with a heavy and a light meson in the final state, if it is the light meson that is emitted from the hard interaction (meson M_2 in Fig. 12). In this case spectator scattering is power suppressed and the factorization theorem in the heavy-quark limit takes the form

$$\langle DM_2 | Q_i | \bar{B} \rangle = F^{B \rightarrow D}(m_2^2) \int_0^1 du T_i^I(u) \Phi_{M_2}(u) \quad (9.6)$$

The expression in (9.6) had already been used in [282] to compute the order- α_s corrections to the ratio of the $\bar{B} \rightarrow D\pi^-$ and $\bar{B} \rightarrow D^*\pi^-$ decay rates, prior to the systematic development of QCD factorization.

The factorization theorem can be formulated using soft-collinear effective theory (SCET) [292, 293]. This formalism is useful for proving factorization [294] and for disentangling the hard and hard-collinear scale in explicit terms. QCD factorization and SCET are theoretical concepts that are fully compatible with each other, but they refer to different aspects of the problem of B -decay matrix elements. In some sense the relation between QCD factorization and SCET is similar to the relation between the heavy-quark expansion (HQE) and heavy-quark effective theory (HQET) in their application to *inclusive* B decays. QCD factorization [279, 280] refers to the separation of the matrix elements into simpler long-distance quantities and calculable hard interactions, where the long-distance form factors are defined in full QCD. SCET, on the other hand, is a general effective field theory formulation for the relevant QCD modes (hard, hard-collinear, collinear, soft) and allows a further separation of scales, for instance in the transition form factors. However, working with form factors in full QCD often seems preferable in practice.

^{††}In the present section the numbering of operators $Q_{1,2}$ and their coefficients is reversed with respect to the notation of the previous sections.

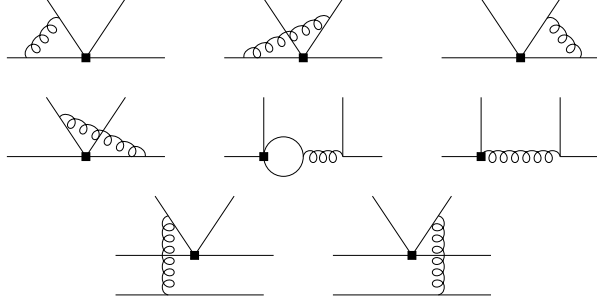


Figure 13: Order α_s corrections to the hard scattering kernels T_i^I (first two rows) and T_i^{II} (last row). In the case of T_i^I , the spectator quark does not participate in the hard interaction and is not drawn.

9.3 NLO calculations

Next-to-leading-logarithmic accuracy (NLO) in the decay amplitudes requires the calculation of the $\mathcal{O}(\alpha_s)$ terms in the factorization formula (9.3). The relevant diagrams for the kernels $T^{I,II}$ are shown in Fig. 13. They provide a concrete illustration of the schematic picture in Fig. 12. As an example, we quote the contribution from current-current operators $Q_{1,2}$ to the $\bar{B} \rightarrow \pi^+ \pi^-$ amplitude. (The penguin contributions are given in [279, 283].) Up to power corrections and to NLO precision this amplitude reads

$$\langle \pi^+ \pi^- | \mathcal{H}_{\text{eff}} | \bar{B} \rangle = i \frac{G_F}{\sqrt{2}} V_{ub} V_{ud}^* F^{B \rightarrow \pi}(0) f_\pi m_B^2 [a_{1,I} + a_{1,II}] + \dots \quad (9.7)$$

where

$$a_{1,I} = C_1 + \frac{C_2}{N} + \frac{C_2 C_F \alpha_s}{N 4\pi} \left[-12 \ln \frac{\mu}{m_b} - 18 + \int_0^1 du \, 3 \left(\frac{1-2u}{1-u} \ln u - i\pi \right) \Phi_\pi(u) \right] \quad (9.8)$$

and ($\bar{u} \equiv 1-u$, $\bar{v} \equiv 1-v$)

$$a_{1,II} = \frac{C_2 C_F \pi \alpha_s}{N N} \frac{f_B f_\pi}{m_B^2 F^{B \rightarrow \pi}(0)} \int_0^1 \frac{d\xi}{\xi} \Phi_B(\xi) \int_0^1 \frac{du}{\bar{u}} \Phi_\pi(u) \int_0^1 \frac{dv}{\bar{v}} \Phi_\pi(v) \quad (9.9)$$

Here $C_F = (N^2 - 1)/(2N)$ and N is the number of colours.

The contribution in (9.8) represents the T^I -part of the matrix element. Coefficients $C_{1,2}$ and α_s are evaluated at a scale $\mu = \mathcal{O}(m_b)$. As it must be the case, the scale and scheme dependence of the α_s -correction cancels against the corresponding dependence of the NLO coefficients up to terms of order α_s^2 . The constant -18 refers to the NDR scheme as defined in [95]. Whereas form factor and decay constant are scheme and scale independent, the pion distribution amplitude Φ_π has such a dependence. Since it enters only at $\mathcal{O}(\alpha_s^2)$ in $a_{1,I}$ this is irrelevant at NLO.

Hard-gluon exchange between the two final-state pions leads to a perturbative rescattering phase and thus to an imaginary part in (9.8). Together with the phase from penguin loops this

gives the formally leading contribution to the rescattering phase in the heavy-quark limit. The numerical value of the imaginary part has to be taken with caution because it has to compete with Λ_{QCD}/m_b power corrections, which are hard to quantify. In any case, the rescattering phase is predicted to be suppressed, either by α_s or by Λ_{QCD}/m_b .

The spectator-scattering term $a_{1,II}$ in (9.9) is an additional, qualitatively different contribution, which first arises at order α_s . The scale dependent quantities C_2 and α_s are evaluated at a scale $\mu_h = \sqrt{\Lambda\mu}$, representing the typical virtuality of the semi-hard (or, more precisely, hard-collinear) gluon in this process.

The factorized structure of the amplitude in (9.7) is in close analogy with the amplitude for other weak processes, for instance B - \bar{B} mixing. All of them consist of certain long-distance quantities multiplying calculable short-distance functions. A technical complication specific to (9.7) is the presence of meson distribution amplitudes, whose factorization involves an integration over parton momentum fractions.

The first calculation of hadronic two-body B -decay amplitudes complete to NLO in QCD was performed in [279] for the three $\bar{B} \rightarrow \pi\pi$ channels $\bar{B}_d \rightarrow \pi^+\pi^-$, $\bar{B}_d \rightarrow \pi^0\pi^0$ and $B^- \rightarrow \pi^-\pi^0$. The class of heavy-light final states $\bar{B}_d \rightarrow D^{(*)+}L^-$, with light meson $L^- = \pi^-, \rho^-, K^{(*)-}, a_1^-, \dots$, was analyzed in detail at NLO in [280] (see also [281]). A recent discussions of phenomenological applications of these modes can be found in [304]. The NLO calculations were subsequently extended to all $B \rightarrow \pi\pi$ and $B \rightarrow \pi K$ channels [283] and eventually to all decays $B \rightarrow PP$ and $B \rightarrow PV$, where P (V) is a light pseudoscalar (vector) meson [284]. Decays into flavour-singlet mesons of the type $B \rightarrow K^{(*)}\eta^{(\prime)}$ and their special properties were treated in [285]. The decays $B \rightarrow VV$ are more complicated due to the existence of different helicity amplitudes for the pair of vector mesons. Only decays into light vector mesons with longitudinal polarization are strictly calculable in QCD factorization. Early papers on this subject are [305, 286]. Comprehensive studies have been given in [287, 288, 289], where [288] also considers final states with axial vector mesons.

The methods of QCD factorization can also be employed for rare and radiative B decays. In this case the dominant part of the amplitude comes from bilinear quark currents, whose matrix elements are directly given by form factors. However, the nonleptonic Hamiltonian contributes to the transition as well and requires a nontrivial application of the factorization formula. The NLO results for the exclusive decays $B \rightarrow K^*\gamma$ and $B \rightarrow \rho\gamma$ were obtained in [241, 225, 306]. This calculation involves the NLO Wilson coefficients for $b \rightarrow s\gamma$ [197], which depend on three-loop anomalous dimensions, and two-loop virtual corrections to the matrix elements of local operators [203, 204]. The work of [241] included the generalization to $B \rightarrow K^*l^+l^-$ at moderate values of the dilepton mass q^2 .

9.4 NNLO calculations

Important progress has been achieved in extending perturbative calculations in QCD factorization for charmless two-body B decays to the NNLO, including effects of order α_s^2 . In many cases this level of accuracy is probably below the size of uncertainties from other sources, in particular from power corrections. However, the explicit knowledge of NNLO corrections is of conceptual interest as it extends the factorization formula to the next nontrivial level in perturbation theory. In addition, there are quantities for which the NNLO effects are likely to be also numerically important. These are cases where a contribution is absent at $\mathcal{O}(1)$ and thus the $\mathcal{O}(\alpha_s)$ term, a NLO contribution in the general counting scheme, is effectively the lowest order. Examples are strong phases relevant for direct CP violation, hard spectator scattering, or the color-suppressed amplitude coefficient a_2 , which is accidentally small at leading and next-to-leading order and therefore rather sensitive to NNLO effects.

We briefly summarize the NNLO corrections that have been computed so far. These are, first, the $\mathcal{O}(\alpha_s^2)$ one-loop hard-spectator interactions^{‡‡} for current-current operators [295, 296, 299] and for penguin contributions [297, 298]. Second, the two-loop vertex corrections (T^I) have been addressed for the first time in [300], where the imaginary part is computed explicitly. The corresponding real part has been obtained in [301], completing the NNLO vertex corrections for current-current operators. These results have been confirmed in [303]. The first phenomenological analysis of exclusive B decays at NNLO has been presented in [302] for the tree-dominated $B \rightarrow \pi\pi$, $\pi\rho$ and $\rho\rho$ decays.

The numerical impact of NNLO effects for the coefficients $a_1(\pi\pi)$ and $a_2(\pi\pi)$, which determine the topological tree-amplitudes in $B \rightarrow \pi\pi$, is illustrated by the following compilation from [301]:

$$\begin{aligned} a_1(\pi\pi) &= 1.008 + [0.022 + 0.009i]_{I,\alpha_s} + [0.024 + 0.026i]_{I,\alpha_s^2} \\ &\quad - [0.012]_{II,\alpha_s} - [0.014 + 0.011i]_{II,\alpha_s^2} - [0.007]_P \\ &= 1.019_{-0.021}^{+0.017} + (0.025_{-0.015}^{+0.019}) i \end{aligned} \tag{9.10}$$

$$\begin{aligned} a_2(\pi\pi) &= 0.224 - [0.174 + 0.075i]_{I,\alpha_s} - [0.030 + 0.048i]_{I,\alpha_s^2} \\ &\quad + [0.075]_{II,\alpha_s} + [0.032 + 0.019i]_{II,\alpha_s^2} + [0.045]_P \\ &= 0.173_{-0.073}^{+0.088} - (0.103_{-0.054}^{+0.051}) i \end{aligned} \tag{9.11}$$

In both equations the first line lists the leading order result together with the vertex corrections (I) at various orders in α_s . Similarly, the second line displays the amplitude from hard-spectator interactions (II). It includes a model-dependent estimate of power corrections (P) from twist-3

^{‡‡}The hard-spectator interactions of $\mathcal{O}(\alpha_s^2)$ are counted as NNLO terms since they enter the decay amplitudes at the NNLO level. A different convention has been used in [295, 297], where the same corrections are referred to as NLO due to the fact that hard-spectator interactions first arise at $\mathcal{O}(\alpha_s)$.

contributions to the light-cone wave function of the pion. The third line shows the full result together with the total uncertainty. The detailed input can be found in [301].

From (9.10) we see that the calculation is well under control and the uncertainties are small. Note that the NLO correction is suppressed due to small Wilson coefficients. In (9.11) the cancellation of leading and next-to-leading vertex contributions is clearly visible. This implies the dominance of the hard spectator term and the relatively large impact of NNLO effects and power corrections. In addition, the first inverse moment of the B -meson distribution amplitude Φ_B , which determines hard-spectator scattering as can be seen in (9.9), is not well known. An accurate prediction of the $B \rightarrow \pi^0 \pi^0$ branching ratio, very sensitive to a_2 , is therefore difficult. The measured number appears to be higher than theoretical estimates. The agreement is better for $B \rightarrow \rho^0 \rho^0$ [302].

We conclude with a few remarks concerning rare and radiative decays. The radiative decays $B \rightarrow V\gamma$ were discussed at NNLO in [227], but some of the contributions needed to complete this order are still missing. Conceptual aspects of the factorization formula for $B \rightarrow V\gamma$ at higher orders in α_s have been treated in [244, 307, 308]. The rare decays $B \rightarrow K^* l^+ l^-$, $B \rightarrow \rho l^+ l^-$ with order- α_s corrections [241, 226], mentioned in 9.3 above, have formally the structure of NNLO processes, requiring the dominant Wilson coefficient C_9 at NNLO [194].

9.5 Outlook

QCD factorization can be applied to a large number of exclusive B decays and the associated phenomenology is very rich. We will not go into any details here, but restrict ourselves to a few remarks. The literature quoted throughout the present section contains much further information on these topics.

Not all observables accessible in principle to a factorization calculation are equally useful in practice. Accurate estimates of power corrections are still beyond our control and effects of typically 10 – 20% are to be expected. This makes it difficult to compute direct CP asymmetries because these are sensitive to the relatively small strong phases, of which only the perturbative (though formally leading) part is calculable. However, there are many cases where the level of precision attainable with QCD factorization provides the basis for accurate predictions and flavour-physics tests.

A first example are suitable ratios of hadronic and semileptonic rates, e.g.

$$\frac{\Gamma(B^- \rightarrow \pi^- \pi^0)}{d\Gamma(\bar{B}_d \rightarrow \pi^+ l^- \bar{\nu})/dq^2|_{q^2=0}} = 3\pi^2 f_\pi^2 |V_{ud}|^2 |a_1 + a_2|^2 \quad (9.12)$$

which are known as factorization tests. While not directly relevant for flavour physics, they test QCD factorization independently of uncertainties from form factors and V_{ub} . At present the precision is still experimentally limited.

The parameter $S(\rho_L^+ \rho_L^-)$ of mixing-induced CP violation in $\bar{B}_d \rightarrow \rho_L^+ \rho_L^-$ is a rather clean quantity. The penguin amplitude is numerically small (even smaller than for $\bar{B}_d \rightarrow \pi^+ \pi^-$)

and can be computed in QCD factorization with little absolute uncertainty. CKM phases may be extracted to within a few degrees [287, 289]. Additional methods to constrain the penguin contribution also benefit from QCD factorization. Using $\bar{B}_d \rightarrow \bar{K}_L^{*0} K_L^{*0}$ and V -spin symmetry should ultimately allow a precise extraction of the CKM angle γ with a theory error of $\pm 1^\circ$ [289].

More generally, QCD factorization can be employed to estimate the size of $SU(3)$ breaking in approaches that rely on flavour symmetries to determine CKM quantities from CP violation in hadronic B decays [309].

Promising observables are exclusive rare and radiative decays such as $B \rightarrow K^* \gamma$, $B \rightarrow \rho \gamma$ or $B \rightarrow K^* l^+ l^-$. They are dominated by form-factor terms, similar to semileptonic modes, but also receive (rather moderate) hadronic contributions. To those the framework of factorization can be successfully applied.

In the upcoming era of precision experiments with B mesons, QCD factorization in the heavy quark limit, at NLO and beyond, provides us with an important tool to control theory predictions at a level adequate for discoveries in flavour physics.

10 Summary

Our story is approaching the end and we reached the summit from which the full field of QCD and QED corrections to weak decays can be seen. As of February 2011 NLO and NNLO QCD and QED corrections to most important decays are known. We collect in Table 7 the NLO and NNLO summits conquered by the MNLC and the names of its members who took part in various expeditions. In Table 8 we collect again all calculations with references to papers where the names of NLO and NNLO climbers outside our club can be found. These efforts increased significantly the accuracy of predictions of the SM for the full field of weak decays. This is an important step towards the indirect searches for New Physics through flavour violating and CP-violating processes. Personally I do not think that the calculations of the still higher orders of perturbation theory are really required and we should wait for the data in order to see what kind of new physics will be identified directly at the LHC and indirectly through flavour violating processes and more generally through high precision experiments. With the technology developed in the last twenty years calculations of higher order QCD corrections in the extensions of the SM selected by Nature should be straightforward even if often tedious. Most of these calculations have been by now automatized as exemplified by a recent paper [310], where further references can be found.

On the other hand still significant progress in non-perturbative calculations is required in order to increase the power of tests of the SM through FCNC processes, in particular in the case of non-leptonic K , D and B decays. While in the case of B -decays, QCDF presented in the previous Section allowed to make an important progress, the case of non-leptonic K -decays

is a different story. In this context I am very curious whether direct CP violation in $K_L \rightarrow \pi\pi$ decays represented by the ratio ε'/ε is well described by the SM or not. Even 22 years after the seminal Ringberg 1988 workshop it was not possible to obtain the prediction for ε'/ε using lattice simulations. Such prediction is, on the other hand possible within the large N approach developed by Bardeen, Gérard and myself in the middle of the 1980's. The most recent analyses of ε'/ε in this framework show that ε'/ε within the SM, although on the lower side, agrees within 2σ with the data [311, 312]. Unfortunately strong cancellations between QCD penguins and electroweak penguins do not allow clear cut conclusions on how much space is still left for New Physics contributions to ε'/ε . We should hope that in this decade the answer will be provided by lattice groups. In this context an interesting paper appeared very recently [313]. Reviews of lattice results can be found in [314, 315, 316].

Presently, much faster development takes place in the case of the parameter \hat{B}_K . Indeed during the last three years an impressive progress in calculating \hat{B}_K has been achieved by means of unquenched lattice simulations. While until recently the values in the ballpark of $\hat{B}_K = 0.724 \pm 0.008 \pm 0.028$ [113] were circulated, the most recent message from RBC and UKQCD collaborations [317] that uses domain wall QCD reads $\hat{B}_K = 0.749 \pm 0.027$.

Interestingly this value is very close to $\hat{B}_K = 0.75$ obtained in the large-N limit of QCD [318, 319]. Including $1/N$ corrections Bardeen, Gérard and myself [4, 320] found some indications for $\hat{B}_K \leq 0.75$. A very recent more precise analysis of Gérard [321] puts this result on firm footing. Thus afterall our large-N result for \hat{B}_K presented already at the 1988 Ringberg workshop and briefly mentioned at the beginning of this writing has been confirmed by much more precise lattice simulations more than 20 years later. In this context also the paper by Bijnens and Prades [322] should be mentioned. See [321] for more details. Whether we were just lucky remains to be seen. In any case the smallness of $1/N$ corrections to the large N value $\hat{B}_K = 0.75$ should be better understood.

I would like to thank all the members of the MNLC for fantastic time we had together conquering the NLO and NNLO summits of weak decays over so many years. In fact after 22 years I can confidently state that my team was much bigger than the one of colonel Hunt and whereas he reached only the south coal on the way to the Mount Everest summit I was lucky to stand on many NLO and few NNLO summits. But similarly to his case the sponsors played also in our case a very important role. In addition to the Physics Department of Technical University Munich, the thanks go primarily to the German ‘Bundesministerium für Bildung und Forschung’ (BMBF) and Deutsche Forschung Gemeinschaft (DFG) which were both very helpful in supporting us for many years. Some support came also from the MPI for Physics and the German-Israeli Foundation. In the last years Martin Gorbahn, Joachim Brod and myself were also strongly supported by the Cluster of Excellence Universe in Munich and the TUM-IAS.

Very special thanks go to Gerhard Buchalla for writing Section 9 of this review. I also thank Paolo Gambino and Matthias Jamin for some help in finding references. Finally, but very

importantly, I would like to thank Guido Martinelli for the discussion at the seminal supper in 1988 in the Ringberg castle and the competition which definitely kept us working for several years. Among the members of his team my particular thanks go to Marco Ciuchini and Luca Silvestrini.

A number of my friends read V1 of this NLO story pointing out misprints and making comments on the full text. These are Gerhard Buchalla, Svjetlana Fajfer, Robert Fleischer, Jean-Marc Gérard, Jennifer Girrbach, Mikolaj Misiak, Matthias Jamin, Cecilia Tarantino, Andreas Weiler and Peter Weisz. I appreciate all these comments and thank them a lot.

References

- [1] W. A. Bardeen, A. J. Buras and J. M. Gérard, “The $\Delta I = 1/2$ Rule in the Large N Limit,” *Phys. Lett. B* **180** (1986) 133.
- [2] W. A. Bardeen, A. J. Buras and J. M. Gérard, “The $K \rightarrow \pi\pi$ Decays in the Large N Limit: Quark Evolution,” *Nucl. Phys. B* **293** (1987) 787.
- [3] W. A. Bardeen, A. J. Buras and J. M. Gérard, “A Consistent Analysis of the $\Delta I = 1/2$ Rule for K Decays,” *Phys. Lett. B* **192** (1987) 138.
- [4] W. A. Bardeen, A. J. Buras and J. M. Gérard, “The B Parameter Beyond The Leading Order of $1/N$ Expansion,” *Phys. Lett. B* **211** (1988) 343.
- [5] G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, “Weak Nonleptonic Decays Beyond Leading Logarithms In QCD,” *Phys. Lett. B* **99** (1981) 141.
- [6] G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, “QCD Nonleading Corrections To Weak Decays As An Application Of Regularization By Dimensional Reduction,” *Nucl. Phys. B* **187** (1981) 461.
- [7] W. Siegel, “Supersymmetric Dimensional Regularization Via Dimensional Reduction,” *Phys. Lett. B* **84** (1979) 193.
- [8] P. Breitenlohner and D. Maison, “Dimensional Renormalization And The Action Principle,” *Commun. Math. Phys.* **52** (1977) 11. P. Breitenlohner and D. Maison, “Dimensionally Renormalized Green’s Functions For Theories With Massless Particles. 1,” *Commun. Math. Phys.* **52** (1977) 39. P. Breitenlohner and D. Maison, “Dimensionally Renormalized Green’s Functions For Theories With Massless Particles. 2,” *Commun. Math. Phys.* **52**, 55 (1977).
- [9] G. ’t Hooft and M. J. G. Veltman, “Regularization And Renormalization Of Gauge Fields,” *Nucl. Phys. B* **44** (1972) 189.

- [10] N. Tracas and N. Vlachos, “Two Loop Calculations In QCD And The $\Delta I = 1/2$ Rule in Nonleptonic Weak Decays,” *Phys. Lett. B* **115** (1982) 419.
- [11] A. J. Buras, J. M. Gerard and W. Huber, “Hadronic matrix elements and weak decays. Proceedings, Ringberg Workshop, Tegernsee, F.R. Germany, April 17-22, 1988,” Amsterdam, Netherlands: North-Holland (1989). *Nucl. Phys. B (Proc. Suppl.)* **7A** (1989) 375 p.
- [12] A. J. Buras, “Phenomenological Applications of the $1/N$ Expansion,” *Nucl. Phys. Proc. Suppl.* **10A** (1989) 199.
- [13] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, “Deep Inelastic Scattering Beyond The Leading Order In Asymptotically Free Gauge Theories,” *Phys. Rev. D* **18** (1978) 3998.
- [14] W. A. Bardeen and A. J. Buras, “Higher Order Asymptotic Freedom Corrections to Photon - Photon Scattering,” *Phys. Rev. D* **20** (1979) 166 [Erratum-ibid. *D* **21** (1980) 2041].
- [15] G. Buchalla, A. J. Buras and M. E. Lautenbacher, “Weak Decays Beyond Leading Logarithms,” *Rev. Mod. Phys.* **68** (1996) 1125 [hep-ph/9512380].
- [16] A. J. Buras, “Weak Hamiltonian, CP Violation and Rare Decays,” in *Probing the Standard Model of Particle Interactions*, R. Gupta, A. Morel, E. de Rafael and F. David eds, Les Houches, Session LXVIII, 1977, Elsevier Science B.V. 1999, page 281. hep-ph/9806471.
- [17] K. G. Wilson, “Nonlagrangian Models Of Current Algebra,” *Phys. Rev.* **179** (1969) 1499.
- [18] W. Zimmermann, “Normal Products And The Short Distance Expansion In The Perturbation Theory Of Renormalizable Interactions,” *Annals Phys.* **77** (1973) 570.
- [19] N. Cabibbo, “Unitary Symmetry And Leptonic Decays,” *Phys. Rev. Lett.* **10** (1963) 531.
- [20] M. Kobayashi and T. Maskawa, “CP Violation In The Renormalizable Theory Of Weak Interaction,” *Prog. Theor. Phys.* **49** (1973) 652.
- [21] D. J. Gross and F. Wilczek, “Ultraviolet Behavior Of Nonabelian Gauge Theories,” *Phys. Rev. Lett.* **30** (1973) 1343.
- [22] H. D. Politzer, “Reliable Perturbative Results For Strong Interactions?,” *Phys. Rev. Lett.* **30** (1973) 1346.
- [23] S. Bethke, “The 2009 Wolrd Average of $\alpha_s(M_Z)$,” *Eur. Phys. J. C* **64** (2009) 689 [arXiv:0908.1135 [hep-ph]].

- [24] S. L. Glashow, J. Iliopoulos and L. Maiani, “Weak Interactions With Lepton - Hadron Symmetry,” *Phys. Rev.* **D 2** (1970) 1285.
- [25] E. C. G. Sudarshan and R. e. Marshak, “Chirality invariance and the universal Fermi interaction,” *Phys. Rev.* **109** (1958) 1860.
- [26] R. P. Feynman and M. Gell-Mann, “Theory of the Fermi interaction,” *Phys. Rev.* **109** (1958) 193.
- [27] W. Buchmuller and D. Wyler, “Effective Lagrangian Analysis Of New Interactions And Flavor Conservation,” *Nucl. Phys. B* **268** (1986) 621.
- [28] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, “Dimension-Six Terms in the Standard Model Lagrangian,” *JHEP* **1010** (2010) 085 [arXiv:1008.4884 [hep-ph]].
- [29] A. J. Buras, M. Misiak and J. Urban, “Two-loop QCD anomalous dimensions of flavour-changing four-quark operators within and beyond the standard model,” *Nucl. Phys. B* **586** (2000) 397 [arXiv:hep-ph/0005183].
- [30] M. Ciuchini, E. Franco, V. Lubicz, G. Martinelli, I. Scimemi and L. Silvestrini, “Next-to-leading order QCD corrections to $\Delta F = 2$ effective Hamiltonians,” *Nucl. Phys. B* **523** (1998) 501 [arXiv:hep-ph/9711402]. M. Ciuchini *et al.*, “ ΔM_K and ε_K in SUSY at the next-to-leading order,” *JHEP* **9810** (1998) 008 [arXiv:hep-ph/9808328].
- [31] A. J. Buras, M. Gorbahn, U. Haisch and U. Nierste, “Charm quark contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ at next-to-next-to-leading order,” *JHEP* **0611** (2006) 002 [arXiv:hep-ph/0603079]. A. J. Buras, M. Gorbahn, U. Haisch and U. Nierste, “The rare decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ at the next-to-next-to-leading order in QCD,” *Phys. Rev. Lett.* **95** (2005) 261805 [arXiv:hep-ph/0508165].
- [32] D. V. Nanopoulos, G. G. Ross, “Rare Decay Modes of the K Mesons and $K_L - K_S$ Mass Difference in an Asymptotically Free Gauge Theory,” *Phys. Lett.* **B56** (1975) 279.
- [33] D. V. Nanopoulos, D. A. Ross, “Limits on the Number of Flavors in Grand Unified Theories from Higher Order Corrections to Fermion Masses,” *Nucl. Phys.* **B157** (1979) 273. R. Tarrach, “The Pole Mass in Perturbative QCD,” *Nucl. Phys.* **B183** (1981) 384.
- [34] O. V. Tarasov, “Anomalous Dimensions Of Quark Masses In Three Loop Approximation,” JINR-P2-82-900. S. A. Larin, “The Renormalization of the axial anomaly in dimensional regularization,” *Phys. Lett.* **B303** (1993) 113-118. [hep-ph/9302240].
- [35] K. G. Chetyrkin, “Quark mass anomalous dimension to $\mathcal{O}(\alpha_s^4)$,” *Phys. Lett.* **B404** (1997) 161-165. [hep-ph/9703278]. J. A. M. Vermaseren, S. A. Larin, T. van Ritbergen, “The

- four loop quark mass anomalous dimension and the invariant quark mass,” Phys. Lett. **B405** (1997) 327-333. [hep-ph/9703284].
- [36] Z. z. Xing, H. Zhang and S. Zhou, “Updated Values of Running Quark and Lepton Masses,” Phys. Rev. D **77** (2008) 113016 [arXiv:0712.1419 [hep-ph]].
- [37] J. Chay, H. Georgi and B. Grinstein, “Lepton Energy Distributions In Heavy Meson Decays From QCD,” Phys. Lett. **B247** (1990) 399. I. I. Bigi, N. G. Uraltsev and A. I. Vainshtein, “Nonperturbative corrections to inclusive beauty and charm decays: QCD versus phenomenological models,” Phys. Lett. **B293** (1992) 430 [hep-ph/9207214]. I. I. Bigi, M. Shifman, N. G. Uraltsev and A. Vainshtein, “QCD predictions for lepton spectra in inclusive heavy flavor decays,” Phys. Rev. Lett. **71** (1993) 496 [hep-ph/9304225]. B. Blok, L. Koyrakh, M. Shifman and A. I. Vainshtein, “Differential distributions in semileptonic decays of the heavy flavors in QCD,” Phys. Rev. D **49** (1994) 3356 [hep-ph/9307247]. A. V. Manohar and M. B. Wise, “Inclusive semileptonic B and polarized Λ_b decays from QCD,” Phys. Rev. D **49** (1994) 1310 [arXiv:hep-ph/9308246]. T. Mannel, “Operator product expansion for inclusive semileptonic decays in heavy quark effective field theory,” Nucl. Phys. **B413** (1994) 396 [hep-ph/9308262].
- [38] M. Neubert, “Heavy quark symmetry,” Phys. Rept. **245** (1994) 259 [hep-ph/9306320].
- [39] A. V. Manohar and M. B. Wise, “Heavy quark physics,” Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10** (2000) 1.
- [40] M. Shifman, “Lectures on heavy quarks in quantum chromodynamics,” hep-ph/9510377.
- [41] A. F. Falk, “The CKM matrix and the heavy quark expansion,” hep-ph/0007339.
- [42] G. Altarelli and L. Maiani, “Octet Enhancement Of Nonleptonic Weak Interactions In Asymptotically Free Gauge Theories,” Phys. Lett. B **52** (1974) 351.
- [43] M. K. Gaillard and B. W. Lee, “ $\Delta I = 1/2$ Rule For Nonleptonic Decays In Asymptotically Free Field Theories,” Phys. Rev. Lett. **33** (1974) 108.
- [44] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Nonleptonic Decays Of K Mesons And Hyperons,” Sov. Phys. JETP **45** (1977) 670.
- [45] F. J. Gilman and M. B. Wise, “Effective Hamiltonian For $\Delta S = 1$ Weak Nonleptonic Decays In The Six Quark Model,” Phys. Rev. D **20** (1979) 2392.
- [46] F. J. Gilman and M. B. Wise, “ $K^0 - \bar{K}^0$ Mixing In The Six Quark Model,” Phys. Rev. D **27** (1983) 1128.

- [47] C. Dib, I. Dunietz and F. J. Gilman, “Strong Interaction Corrections To The Decay $K \rightarrow \pi\nu\bar{\nu}$ For Large m_t ,” Mod. Phys. Lett. A **6** (1991) 3573.
- [48] G. Buchalla and A. J. Buras, “QCD corrections to the $\bar{s}dZ$ vertex for arbitrary top quark mass,” Nucl. Phys. B **398** (1993) 285.
- [49] G. Buchalla and A. J. Buras, “QCD corrections to rare K and B decays for arbitrary top quark mass,” Nucl. Phys. B **400** (1993) 225.
- [50] A. J. Buras, S. Jäger and J. Urban, “Master formulae for $\Delta F = 2$ NLO-QCD factors in the standard model and beyond,” Nucl. Phys. B **605** (2001) 600 [arXiv:hep-ph/0102316].
- [51] G. Buchalla and A. J. Buras, “The rare decays $K^+ \rightarrow \pi^+\nu\bar{\nu}$ and $K_L \rightarrow \mu^+\mu^-$ beyond leading logarithms,” Nucl. Phys. B **412** (1994) 106 [arXiv:hep-ph/9308272].
- [52] M. Gorbahn and U. Haisch, “Charm quark contribution to $K_L \rightarrow \mu^+\mu^-$ at next-to-next-to-leading order,” Phys. Rev. Lett. **97** (2006) 122002 [arXiv:hep-ph/0605203].
- [53] A. J. Buras, P. Gambino and U. A. Haisch, “Electroweak penguin contributions to non-leptonic $\Delta F = 1$ decays at NNLO,” Nucl. Phys. B **570** (2000) 117 [arXiv:hep-ph/9911250].
- [54] C. Bobeth, M. Misiak and J. Urban, “Photonic penguins at two loops and m_t -dependence of $\text{Br}(B \rightarrow X_s l^+ l^-)$,” Nucl. Phys. B **574** (2000) 291 [arXiv:hep-ph/9910220].
- [55] C. Bobeth, M. Misiak and J. Urban, “Matching conditions for $b \rightarrow s\gamma$ and $b \rightarrow s$ gluon in extensions of the standard model,” Nucl. Phys. B **567** (2000) 153 [arXiv:hep-ph/9904413].
- [56] G. Buchalla and A. J. Buras, “Two-loop large- m_t electroweak corrections to $K \rightarrow \pi\nu\bar{\nu}$ for arbitrary Higgs boson mass,” Phys. Rev. D **57** (1998) 216 [arXiv:hep-ph/9707243].
- [57] J. Brod, M. Gorbahn and E. Stamou, “Two-Loop Electroweak Corrections for the $K - > \pi\nu n\bar{u}$ Decays,” Phys. Rev. D **83** (2011) 034030 [arXiv:1009.0947 [hep-ph]].
- [58] P. Gambino, A. Kwiatkowski and N. Pott, “Electroweak effects in the $B^0 - \bar{B}^0$ mixing,” Nucl. Phys. B **544** (1999) 532 [arXiv:hep-ph/9810400].
- [59] A. Czarnecki and W. J. Marciano, “Electroweak radiative corrections to $b \rightarrow s\gamma$,” Phys. Rev. Lett. **81** (1998) 277 [arXiv:hep-ph/9804252]. K. Baranowski and M. Misiak, “The $\mathcal{O}(\alpha\alpha_s)$ correction to $\text{Br}(B \rightarrow X_s\gamma)$,” Phys. Lett. B **483** (2000) 410 [arXiv:hep-ph/9907427].
- [60] A. Strumia, “Two loop heavy top corrections to the $b \rightarrow s\gamma$ decay,” Nucl. Phys. B **532** (1998) 28 [arXiv:hep-ph/9804274].

- [61] A. L. Kagan and M. Neubert, “QCD anatomy of $B \rightarrow X_s \gamma$ decays,” *Eur. Phys. J. C* **7** (1999) 5 [arXiv:hep-ph/9805303].
- [62] P. Gambino and U. Haisch, “Electroweak effects in radiative B decays,” *JHEP* **0009** (2000) 001 [arXiv:hep-ph/0007259].
- [63] P. Gambino and U. Haisch, “Complete electroweak matching for radiative B decays,” *JHEP* **0110** (2001) 020 [arXiv:hep-ph/0109058].
- [64] J. Urban, F. Krauss, U. Jentschura and G. Soff, “Next-to-leading order QCD corrections for the $B^0 - \bar{B}^0$ mixing with an extended Higgs sector,” *Nucl. Phys. B* **523** (1998) 40 [arXiv:hep-ph/9710245].
- [65] F. Borzumati and C. Greub, “2HDMs predictions for $B \rightarrow X_s \gamma$ in NLO QCD,” *Phys. Rev. D* **58** (1998) 074004 [arXiv:hep-ph/9802391]. F. M. Borzumati and C. Greub, “Two Higgs doublet model predictions for $B \rightarrow X_s \gamma$ in NLO QCD. (Addendum),” *Phys. Rev. D* **59** (1999) 057501 [arXiv:hep-ph/9809438].
- [66] F. Borzumati, C. Greub and Y. Yamada, “Beyond leading-order corrections to $B \rightarrow X_s \gamma$ at large $\tan \beta$: The charged-Higgs contribution,” *Phys. Rev. D* **69** (2004) 055005 [arXiv:hep-ph/0311151].
- [67] M. Ciuchini, G. Degrandi, P. Gambino and G. F. Giudice, “Next-to-leading QCD corrections to $B \rightarrow X_s \gamma$: Standard model and two-Higgs doublet model,” *Nucl. Phys. B* **527** (1998) 21 [arXiv:hep-ph/9710335].
- [68] P. Ciafaloni, A. Romanino and A. Strumia, “Two-loop QCD corrections to charged-Higgs-mediated $b \rightarrow s \gamma$ decay,” *Nucl. Phys. B* **524** (1998) 361 [arXiv:hep-ph/9710312].
- [69] S. Schilling, C. Greub, N. Salzmann and B. Toedtli, “QCD corrections to the Wilson coefficients C_9 and C_{10} in two-Higgs-doublet models,” *Phys. Lett. B* **616** (2005) 93 [arXiv:hep-ph/0407323].
- [70] T. Hermann, M. Misiak and M. Steinhauser, “ $\bar{B} \rightarrow X_s \gamma$ in the Two Higgs Doublet Model up to Next-to-Next-to-Leading Order in QCD,” arXiv:1208.2788 [hep-ph].
- [71] M. Ciuchini, E. Franco, D. Guadagnoli, V. Lubicz, V. Porretti and L. Silvestrini, “NLO calculation of the $\Delta F = 2$ Hamiltonians in the MSSM and phenomenological analysis of the $B^0 - \bar{B}^0$ mixing,” *PoS HEP2005* (2006) 221 [arXiv:hep-ph/0512141].
- [72] E. Bertuzzo, M. Farina and P. Lodone, “On the QCD corrections to Delta F=2 FCNC in the Supersymmetric SM with hierarchical squark masses,” *Phys. Lett. B* **699** (2011) 98 [arXiv:1011.3240 [hep-ph]].

- [73] J. Virto, “Exact NLO strong interaction corrections to the Delta F=2 effective Hamiltonian in the MSSM,” JHEP **0911** (2009) 055 [arXiv:0907.5376 [hep-ph]]. J. Virto, “NLO corrections to the Delta F=2 Hamiltonian in the MSSM with non-degenerate squarks,” AIP Conf. Proc. **1200** (2010) 875 [arXiv:0909.3277 [hep-ph]].
- [74] J. Virto, “Top mass dependent $O(\alpha_s^3)$ corrections to B-meson mixing in the MSSM,” JHEP **1201** (2012) 120 [arXiv:1111.0940 [hep-ph]].
- [75] F. Borzumati, C. Greub, T. Hurth and D. Wyler, “Gluino contribution to radiative B decays: Organization of QCD corrections and leading order results,” Phys. Rev. D **62** (2000) 075005 [arXiv:hep-ph/9911245].
- [76] T. F. Feng, X. Q. Li, W. G. Ma and F. Zhang, “Complete analysis on the NLO SUSY-QCD corrections to $B^0 - \bar{B}^0$ mixing,” Phys. Rev. D **63** (2001) 015013 [arXiv:hep-ph/0008029].
- [77] M. Ciuchini, G. Degrassi, P. Gambino and G. F. Giudice, “Next-to-leading QCD corrections to $B \rightarrow X_s \gamma$ in supersymmetry,” Nucl. Phys. B **534** (1998) 3 [arXiv:hep-ph/9806308].
- [78] C. Greub, T. Hurth, V. Pilipp, C. Schupbach and M. Steinhauser, “Complete next-to-leading order gluino contributions to $b \rightarrow s \gamma$ and $b \rightarrow s \text{gluon}$,” Nucl. Phys. B **853** (2011) 240 [arXiv:1105.1330 [hep-ph]].
- [79] C. Greub, T. Hurth, V. Pilipp and C. Schuepbach, “Perturbative Corrections to $B \rightarrow X_s \gamma$ in Supersymmetry at Next-to-Leading Order,” arXiv:1111.3692 [hep-ph].
- [80] C. Bobeth, A. J. Buras, F. Kruger and J. Urban, “QCD corrections to $B \rightarrow X_{d,s} \nu \bar{\nu}$, $B_{d,s} \rightarrow l^+ l^-$, $K \rightarrow \pi \nu \bar{\nu}$ and $K_L \rightarrow \mu^+ \mu^-$ in the MSSM,” Nucl. Phys. B **630** (2002) 87 [arXiv:hep-ph/0112305].
- [81] C. Bobeth, A. J. Buras and T. Ewerth, “ $B \rightarrow X_s l^+ l^-$ in the MSSM at NNLO,” Nucl. Phys. B **713** (2005) 522 [arXiv:hep-ph/0409293].
- [82] T. Inami and C. S. Lim, “Effects Of Superheavy Quarks And Leptons In Low-Energy Weak Processes $K_L \rightarrow \mu^+ \mu^-$, $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ And $K^0 - \bar{K}^0$,” Prog. Theor. Phys. **65** (1981) 297.
- [83] G. Buchalla, A. J. Buras and M. K. Harlander, “Penguin box expansion: Flavor changing neutral current processes and a heavy top quark,” Nucl. Phys. **B349** (1991) 1.
- [84] A. J. Buras, “Minimal flavour violation and beyond: Towards a flavour code for short distance dynamics,” Acta Phys. Polon. B **41** (2010) 2487 [arXiv:1012.1447 [hep-ph]].

- [85] A. J. Buras, “Minimal flavor violation,” *Acta Phys. Polon. B* **34** (2003) 5615 [arXiv:hep-ph/0310208].
- [86] A. J. Buras, M. V. Carlucci, S. Gori and G. Isidori, “Higgs-mediated FCNCs: Natural Flavour Conservation vs. Minimal Flavour Violation,” *JHEP* **1010** (2010) 009 [arXiv:1005.5310 [hep-ph]].
- [87] A. J. Buras, G. Isidori and P. Paradisi, “EDMs vs. CPV in $B_{s,d}$ mixing in two Higgs doublet models with MFV,” *Phys. Lett. B* **694** (2011) 402 [arXiv:1007.5291 [hep-ph]].
- [88] A. J. Buras, K. Gemmler and G. Isidori, “Quark flavour mixing with right-handed currents: an effective theory approach,” *Nucl. Phys. B* **843** (2011) 107 [arXiv:1007.1993 [hep-ph]].
- [89] A. J. MacFarlane, A. Sudbery and P. H. Weisz, “On Gell-Mann’s Gamma Matrices, D Tensors And F Tensors, Octets, And Parametrizations Of SU(3),” *Commun. Math. Phys.* **11** (1968) 77.
- [90] A. J. Buras and P. H. Weisz, “QCD Nonleading Corrections To Weak Decays In Dimensional Regularization and ’t Hooft-Veltman Schemes,” *Nucl. Phys. B* **333** (1990) 66.
- [91] M. J. Dugan and B. Grinstein, “On the vanishing of evanescent operators,” *Phys. Lett. B* **256** (1991) 239.
- [92] S. Herrlich and U. Nierste, “Evanescent operators, scheme dependences and double insertions,” *Nucl. Phys. B* **455** (1995) 39 [arXiv:hep-ph/9412375].
- [93] M. Gorbahn and U. Haisch, “Effective Hamiltonian for non-leptonic $\Delta F = 1$ decays at NNLO in QCD,” *Nucl. Phys. B* **713** (2005) 291 [arXiv:hep-ph/0411071].
- [94] A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, “Effective Hamiltonians for $\Delta S = 1$ and $\Delta B = 1$ nonleptonic decays beyond the leading logarithmic approximation,” *Nucl. Phys. B* **370** (1992) 69 [Addendum-ibid. *B* **375** (1992) 501].
- [95] A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, “Two loop anomalous dimension matrix for $\Delta S = 1$ weak nonleptonic decays. 1. $\mathcal{O}(\alpha_s^2)$,” *Nucl. Phys. B* **400** (1993) 37 [arXiv:hep-ph/9211304].
- [96] A. J. Buras, M. Jamin and M. E. Lautenbacher, “The Anatomy of ε'/ε beyond leading logarithms with improved hadronic matrix elements,” *Nucl. Phys. B* **408** (1993) 209 [arXiv:hep-ph/9303284].
- [97] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, “ ε'/ε at the Next-to-leading order in QCD and QED,” *Phys. Lett. B* **301** (1993) 263 [arXiv:hep-ph/9212203]. M. Ciuchini,

- E. Franco, G. Martinelli, L. Reina and L. Silvestrini, “An Upgraded analysis of ε'/ε at the next-to-leading order,” *Z. Phys. C* **68** (1995) 239 [arXiv:hep-ph/9501265].
- [98] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, “The $\Delta S = 1$ effective Hamiltonian including next-to-leading order QCD and QED corrections,” *Nucl. Phys. B* **415** (1994) 403 [arXiv:hep-ph/9304257].
- [99] K. G. Chetyrkin, M. Misiak and M. Münz, “ $|\Delta(F)| = 1$ nonleptonic effective Hamiltonian in a simpler scheme,” *Nucl. Phys. B* **520** (1998) 279 [arXiv:hep-ph/9711280].
- [100] R. Fleischer, “CP violating asymmetries in penguin induced B meson decays beyond the leading logarithmic approximation,” *Z. Phys. C* **58** (1993) 483. R. Fleischer, “Electroweak Penguin effects beyond leading logarithms in the B meson decays $B^- \rightarrow K^- \phi$ and $B^- \rightarrow \pi^- \bar{K}^0$,” *Z. Phys. C* **62** (1994) 81. R. Fleischer, “CP violation and the role of electroweak penguins in nonleptonic B decays,” *Int. J. Mod. Phys. A* **12** (1997) 2459 [arXiv:hep-ph/9612446].
- [101] A. J. Buras, M. Jamin and M. E. Lautenbacher, “Two loop anomalous dimension matrix for $\Delta S = 1$ weak nonleptonic decays. 2. $\mathcal{O}(\alpha_s)$,” *Nucl. Phys. B* **400** (1993) 75 [arXiv:hep-ph/9211321].
- [102] E. Bagan, P. Ball, V. M. Braun and P. Gosdzinsky, “Theoretical update of the semileptonic branching ratio of B mesons,” *Phys. Lett. B* **342** (1995) 362 [Erratum-ibid. B **374** (1996) 363] [arXiv:hep-ph/9409440].
- [103] A. Lenz, U. Nierste and G. Ostermaier, “Penguin diagrams, charmless B decays and the *missing charm puzzle*,” *Phys. Rev. D* **56** (1997) 7228 [arXiv:hep-ph/9706501].
- [104] A. Lenz, U. Nierste and G. Ostermaier, “Determination of the CKM angle γ and $|V(ub)/V(cb)|$ from inclusive direct CP asymmetries and branching ratios in charmless B decays,” *Phys. Rev. D* **59** (1999) 034008 [arXiv:hep-ph/9802202].
- [105] C. Greub, M. Neubert and B. D. Pecjak, “NNLO corrections to $B \rightarrow X_u l \nu$ and the determination of $|V_{ub}|$,” *Eur. Phys. J. C* **65** (2010) 501 [arXiv:0909.1609 [hep-ph]].
- [106] G. Buchalla, “ $\mathcal{O}(\alpha_s)$ QCD corrections to charm quark decay in dimensional regularization with nonanticommuting γ_5 ,” *Nucl. Phys. B* **391** (1993) 501.
- [107] E. Bagan, P. Ball, V. M. Braun and P. Gosdzinsky, “Charm quark mass dependence of QCD corrections to nonleptonic inclusive B decays,” *Nucl. Phys. B* **432** (1994) 3 [arXiv:hep-ph/9408306].
- [108] E. Bagan, P. Ball, B. Fiol and P. Gosdzinsky, “Next-to-leading order radiative corrections to the decay $b \rightarrow ccs$,” *Phys. Lett. B* **351** (1995) 546 [arXiv:hep-ph/9502338].

- [109] M. Jamin and A. Pich, “QCD corrections to inclusive $\Delta S = 1, 2$ transitions at the next-to-leading order,” Nucl. Phys. **B425** (1994) 15 [hep-ph/9402363].
- [110] M. Jamin and M. E. Lautenbacher, “TRACER: Version 1.1: A Mathematica package for gamma algebra in arbitrary dimensions,” Comput. Phys. Commun. **74** (1993) 265.
- [111] A. J. Buras, “Asymptotic Freedom In Deep Inelastic Processes In The Leading Order And Beyond,” Rev. Mod. Phys. **52** (1980) 199.
- [112] A. J. Buras, M. Jamin, M. E. Lautenbacher and P. H. Weisz, “Penguin contributions to nonleptonic decays beyond the leading logarithmic approximation,” in Proc. of Lepton-Photon Conf., Geneva, Switzerland, Jul 25 - Aug 1, 1991. Conf.Proc.C910725V1:199-201,1991.
- [113] D. J. Antonio *et al.* [RBC Collaboration and UKQCD Collaboration], “Neutral kaon mixing from 2+1 flavor domain wall QCD,” Phys. Rev. Lett. **100** (2008) 032001 [arXiv:hep-ph/0702042]. C. Aubin, J. Laiho and R. S. Van de Water, “The neutral kaon mixing parameter B_K from unquenched mixed-action lattice QCD,” Phys. Rev. D **81** (2010) 014507 [arXiv:0905.3947 [hep-lat]].
- [114] T. Bae, Y. -C. Jang, C. Jung, H. -J. Kim, J. Kim, K. Kim, W. Lee and S. R. Sharpe *et al.*, “ B_K using HYP-smearred staggered fermions in $N_f = 2 + 1$ unquenched QCD,” Phys. Rev. D **82** (2010) 114509 [arXiv:1008.5179 [hep-lat]].
- [115] M. Constantinou *et al.* [ETM Collaboration], “ B_K -parameter from $N_f = 2$ twisted mass lattice QCD,” Phys. Rev. D **83** (2011) 014505 [arXiv:1009.5606 [hep-lat]]. Y. Aoki, R. Arthur, T. Blum, P. A. Boyle, D. Brommel, N. H. Christ, C. Dawson and T. Izubuchi *et al.*, “Continuum Limit of B_K from 2+1 Flavor Domain Wall QCD,” Phys. Rev. D **84** (2011) 014503 [arXiv:1012.4178 [hep-lat]].
- [116] J. Kim, C. Jung, H. -J. Kim, W. Lee and S. R. Sharpe, Phys. Rev. D **83** (2011) 117501 [arXiv:1101.2685 [hep-lat]].
- [117] A. J. Buras and M. E. Lautenbacher, “An Analytic formula and an upper bound for ε'/ε in the Standard Model,” Phys. Lett. B **318** (1993) 212 [arXiv:hep-ph/9309203].
- [118] A. J. Buras, M. Jamin and M. E. Lautenbacher, “A 1996 analysis of the CP violating ratio ε'/ε ,” Phys. Lett. B **389** (1996) 749 [arXiv:hep-ph/9608365].
- [119] A. J. Buras and M. Jamin, “ ε'/ε at the NLO: 10 years later,” JHEP **0401** (2004) 048 [arXiv:hep-ph/0306217].
- [120] V. Aquila, P. Gambino, G. Ridolfi and N. Uraltsev, “Perturbative corrections to semileptonic b decay distributions,” Nucl. Phys. B **719** (2005) 77 [arXiv:hep-ph/0503083].

- P. Gambino and C. Schwanda, “Theoretical and Experimental Status of Inclusive Semileptonic Decays and Fits for $|V_{cb}|$,” arXiv:1102.0210 [hep-ex].
- [121] A. Ali and E. Pietarinen, “Semileptonic Decays Of Heavy Quarks In Quantum Chromodynamics,” Nucl. Phys. B **154** (1979) 519; G. Altarelli, N. Cabibbo, G. Corbo, L. Maiani and G. Martinelli, “Leptonic Decay Of Heavy Flavors: A Theoretical Update,” Nucl. Phys. B **208** (1982) 365.
- [122] M. Jezabek and J. H. Kuhn, “QCD Corrections To Semileptonic Decays Of Heavy Quarks,” Nucl. Phys. B **314** (1989) 1; “Lepton Spectra From Heavy Quark Decay,” Nucl. Phys. B **320** (1989) 20; A. Czarnecki, M. Jezabek and J. H. Kuhn, “Hadron Spectra From Semileptonic Decays Of Heavy Quarks,” Acta Phys. Polon. B **20** (1989) 961; A. Czarnecki and M. Jezabek, “Distributions of leptons in decays of polarized heavy quarks,” Nucl. Phys. B **427** (1994) 3 [arXiv:hep-ph/9402326].
- [123] Y. Nir, “The Mass Ratio m_c/m_b in Semileptonic B Decays,” Phys. Lett. B **221** (1989) 184.
- [124] A. F. Falk, M. E. Luke and M. J. Savage, “Hadron spectra for semileptonic heavy quark decay,” Phys. Rev. **D53** (1996) 2491. A. F. Falk, M. E. Luke, “Hadronic spectral moments in semileptonic b decays with a lepton energy cut,” Phys. Rev. **D57** (1998) 424-430. [hep-ph/9708327].
- [125] M. Trott, “Improving extractions of $|V_{cb}|$ and m_b from the hadronic invariant mass moments of semileptonic inclusive B decay,” Phys. Rev. D **70** (2004) 073003 [arXiv:hep-ph/0402120].
- [126] V. Aquila, P. Gambino, G. Ridolfi and N. Uraltsev, “Perturbative corrections to semileptonic b decay distributions,” Nucl. Phys. B **719** (2005) 77 [arXiv:hep-ph/0503083].
- [127] M. E. Luke, M. J. Savage and M. B. Wise, “Charm mass dependence of the $\mathcal{O}(\alpha_s^2 n_f)$ correction to inclusive $B \rightarrow X_c e \bar{\nu}_e$ decay,” Phys. Lett. B **345** (1995) 301 [arXiv:hep-ph/9410387].
- [128] P. Ball, M. Beneke and V. M. Braun, “Resummation of running coupling effects in semileptonic B meson decays and extraction of $|V_{cb}|$,” Phys. Rev. D **52** (1995) 3929 [arXiv:hep-ph/9503492].
- [129] M. Gremm and I. Stewart, “Order $\alpha_s^2 \beta_0$ correction to the charged lepton spectrum in $b \rightarrow c l \bar{\nu}_l$ decays,” Phys. Rev. **D55** (1997) 1226.
- [130] A. Pak and A. Czarnecki, “Mass effects in muon and semileptonic $b \rightarrow c$ decays,” Phys. Rev. Lett. **100** (2008) 241807 [arXiv:0803.0960 [hep-ph]].

- [131] K. Melnikov, “ $\mathcal{O}(\alpha_s^2)$ corrections to semileptonic decay $b \rightarrow cl\bar{\nu}_l$,” Phys. Lett. B **666** (2008) 336 [arXiv:0803.0951 [hep-ph]]. S. Biswas and K. Melnikov, “Second order QCD corrections to inclusive semileptonic $b \rightarrow X_c l \bar{\nu}_l$ decays with massless and massive lepton,” JHEP **1002** (2010) 089 [arXiv:0911.4142 [hep-ph]].
- [132] A. Czarnecki, “Two-loop QCD corrections to $b \rightarrow c$ transitions at zero recoil,” Phys. Rev. Lett. **76** (1996) 4124 [arXiv:hep-ph/9603261]; A. Czarnecki and K. Melnikov, “Two-loop QCD corrections to semileptonic b decays at maximal recoil,” Phys. Rev. Lett. **78** (1997) 3630 [arXiv:hep-ph/9703291]; A. Czarnecki and K. Melnikov, “Two-loop QCD corrections to semileptonic b decays at an intermediate recoil,” Phys. Rev. D **59** (1999) 014036 [arXiv:hep-ph/9804215].
- [133] T. van Ritbergen, “The second order QCD contribution to the semileptonic $b \rightarrow u$ decay rate,” Phys. Lett. B **454** (1999) 353 [arXiv:hep-ph/9903226].
- [134] F. De Fazio and M. Neubert, “ $B \rightarrow X_u l \bar{\nu}_l$ decay distributions to order $\alpha(s)$,” JHEP **9906**, 017 (1999) [arXiv:hep-ph/9905351].
- [135] P. Gambino, E. Gardi and G. Ridolfi, “Running-coupling effects in the triple-differential charmless semileptonic decay width,” JHEP **0612** (2006) 036 [arXiv:hep-ph/0610140].
- [136] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, “On The Motion Of Heavy Quarks Inside Hadrons: Universal Distributions And Inclusive Decays,” Int. J. Mod. Phys. A **9** (1994) 2467 [arXiv:hep-ph/9312359]. M. Neubert, “QCD based interpretation of the lepton spectrum in inclusive $\bar{B} \rightarrow X_u l \bar{\nu}_l$ decays,” Phys. Rev. **D49**, 3392-3398 (1994). [hep-ph/9311325]. S. W. Bosch, B. O. Lange, M. Neubert *et al.*, “Factorization and shape function effects in inclusive B meson decays,” Nucl. Phys. **B699** (2004) 335-386. [hep-ph/0402094]. J. R. Andersen and E. Gardi, “Inclusive spectra in charmless semileptonic B decays by dressed gluon exponentiation,” JHEP **0601** (2006) 097 [arXiv:hep-ph/0509360].
- [137] A. K. Leibovich, Z. Ligeti and M. B. Wise, “Enhanced subleading structure functions in semileptonic B decay,” Phys. Lett. B **539** (2002) 242 [arXiv:hep-ph/0205148]; C. W. Bauer, M. Luke and T. Mannel, “Subleading shape functions in $B \rightarrow X_u l \nu_l$ and the determination of $|V_{cb}|$,” Phys. Lett. B **543** (2002) 261 [arXiv:hep-ph/0205150]; S. W. Bosch, M. Neubert and G. Paz, “Subleading shape functions in inclusive B decays,” JHEP **0411** (2004) 073 [arXiv:hep-ph/0409115]. P. Gambino, P. Giordano, G. Ossola and N. Uraltsev, “Inclusive semileptonic B decays and the determination of $|V_{ub}|$,” JHEP **0710** (2007) 058 [arXiv:0707.2493 [hep-ph]].
- [138] H. M. Asatrian, C. Greub and B. D. Pecjak, “NNLO corrections to $B \rightarrow X_u l \nu$ in the shape-function region,” Phys. Rev. D **78** (2008) 114028 [arXiv:0810.0987 [hep-ph]].

- R. Bonciani, A. Ferroglia, “Two-Loop QCD Corrections to the Heavy-to-Light Quark Decay,” JHEP **0811** (2008) 065. [arXiv:0809.4687 [hep-ph]].
- [139] M. Beneke, T. Huber and X. Q. Li, “Two-loop QCD correction to differential semi-leptonic $b \rightarrow u$ decays in the shape-function region,” Nucl. Phys. B **811** (2009) 77 [arXiv:0810.1230 [hep-ph]].
- [140] A. J. Buras, W. Slominski and H. Steger, “B Meson Decay, CP Violation, Mixing Angles And The Top Quark Mass,” Nucl. Phys. B **238** (1984) 529.
- [141] A. J. Buras, W. Slominski and H. Steger, “ $B^0 - \bar{B}^0$ Mixing, CP Violation And The B Meson Decay,” Nucl. Phys. B **245** (1984) 369.
- [142] A. J. Buras, M. Jamin and P. H. Weisz, “Leading And Next-To-Leading QCD Corrections to ε Parameter and $B^0 - \bar{B}^0$ Mixing in the Presence of A Heavy Top Quark,” Nucl. Phys. B **347** (1990) 491.
- [143] S. Herrlich and U. Nierste, “Enhancement of the $K_L - K_S$ mass difference by short distance QCD corrections beyond leading logarithms,” Nucl. Phys. B **419** (1994) 292 [arXiv:hep-ph/9310311].
- [144] S. Herrlich and U. Nierste, “Indirect CP violation in the neutral kaon system beyond leading logarithms,” Phys. Rev. D **52** (1995) 6505 [arXiv:hep-ph/9507262].
- [145] S. Herrlich and U. Nierste, “The Complete $|\Delta S| = 2$ Hamiltonian in the Next-To-Leading Order,” Nucl. Phys. B **476** (1996) 27 [arXiv:hep-ph/9604330].
- [146] J. Brod and M. Gorbahn, “ ε_K at NNLO: The Charm-Top-Quark Contribution,” Phys. Rev. D **82** (2010) 094026 [arXiv:1007.0684 [hep-ph]].
- [147] J. Brod and M. Gorbahn, “Next-to-Next-to-Leading-Order Charm-Quark Contribution to the CP Violation Parameter ε_K and ΔM_K ,” Phys. Rev. Lett. **108** (2012) 121801 [arXiv:1108.2036 [hep-ph]].
- [148] A. J. Buras and D. Guadagnoli, “Correlations among new CP violating effects in $\Delta F = 2$ observables,” Phys. Rev. D **78**, 033005 (2008) [arXiv:0805.3887 [hep-ph]].
- [149] A. J. Buras, D. Guadagnoli and G. Isidori, “On ε_K beyond lowest order in the Operator Product Expansion,” Phys. Lett. B **688** (2010) 309 [arXiv:1002.3612 [hep-ph]].
- [150] M. Beneke, G. Buchalla, C. Greub, A. Lenz and U. Nierste, “Next-to-leading order QCD corrections to the lifetime difference of B_s mesons,” Phys. Lett. B **459** (1999) 631 [arXiv:hep-ph/9808385].

- [151] M. Ciuchini, E. Franco, V. Lubicz and F. Mescia, “Next-to-leading order QCD corrections to spectator effects in lifetimes of beauty hadrons,” Nucl. Phys. B **625** (2002) 211 [arXiv:hep-ph/0110375].
- [152] M. Ciuchini, E. Franco, V. Lubicz, F. Mescia and C. Tarantino, “Lifetime differences and CP violation parameters of neutral B mesons at the next-to-leading order in QCD,” JHEP **0308** (2003) 031 [arXiv:hep-ph/0308029].
- [153] M. Beneke, G. Buchalla, C. Greub, A. Lenz and U. Nierste, “The $B^+ - B_d^0$ lifetime difference beyond leading logarithms,” Nucl. Phys. B **639** (2002) 389 [arXiv:hep-ph/0202106].
- [154] M. Beneke, G. Buchalla, A. Lenz and U. Nierste, “CP asymmetry in flavour-specific B decays beyond leading logarithms,” Phys. Lett. B **576** (2003) 173 [arXiv:hep-ph/0307344].
- [155] A. Lenz and U. Nierste, “Theoretical update of $B_s - \bar{B}_s$ mixing,” JHEP **0706** (2007) 072 [arXiv:hep-ph/0612167]. A. Lenz and U. Nierste; “Numerical updates of lifetimes and mixing parameters of B mesons,” arXiv:1102.4274 [hep-ph].
- [156] A. J. Buras and J. Girrbach, “Complete NLO QCD Corrections for Tree Level $\Delta F = 2$ FCNC Processes,” JHEP **1203** (2012) 052 [arXiv:1201.1302 [hep-ph]].
- [157] A. J. Buras and J. Girrbach, “Completing NLO QCD Corrections for Tree Level Non-Leptonic $\Delta F = 1$ Decays Beyond the Standard Model,” JHEP **1202** (2012) 143 [arXiv:1201.2563 [hep-ph]].
- [158] A. J. Buras, F. Schwab and S. Uhlig, “Waiting for precise measurements of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K_L \rightarrow \pi^0 \nu \bar{\nu}$,” Rev. Mod. Phys. **80** (2008) 965 [arXiv:hep-ph/0405132].
- [159] J. Fleischer, O. V. Tarasov, F. Jegerlehner and P. Raczka, “Two Loop $\mathcal{O}(\alpha_s G_\mu m_t^2)$ Corrections To The Partial Decay Width of the $Z^0 \rightarrow b\bar{b}$ final States In The Large Top Mass Limit,” Phys. Lett. B **293** (1992) 437.
- [160] M. Misiak and J. Urban, “QCD corrections to FCNC decays mediated by Z-penguins and W-boxes,” Phys. Lett. B **451** (1999) 161 [arXiv:hep-ph/9901278].
- [161] G. Buchalla and A. J. Buras, “The rare decays $K \rightarrow \pi \nu \bar{\nu}$, $B \rightarrow X \nu \bar{\nu}$ and $B \rightarrow l^+ l^-$: An update,” Nucl. Phys. B **548** (1999) 309 [arXiv:hep-ph/9901288].
- [162] G. Buchalla and G. Isidori, “The CP conserving contribution to $K_L \rightarrow \pi^0 \nu \bar{\nu}$ in the standard model,” Phys. Lett. B **440** (1998) 170 [arXiv:hep-ph/9806501].
- [163] G. Isidori, F. Mescia and C. Smith, “Light-quark loops in $K \rightarrow \pi \nu \bar{\nu}$,” Nucl. Phys. B **718** (2005) 319 [arXiv:hep-ph/0503107].

- [164] F. Mescia and C. Smith, “Improved estimates of rare K decay matrix-elements from $K(l3)$ decays,” Phys. Rev. D **76** (2007) 034017 [arXiv:0705.2025 [hep-ph]].
- [165] G. Buchalla and A. J. Buras, “Parity violating longitudinal muon polarization in $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ beyond leading logarithms,” Phys. Lett. B **336** (1994) 263 [arXiv:hep-ph/9407342].
- [166] J. Brod and M. Gorbahn, “Electroweak Corrections to the Charm Quark Contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$,” Phys. Rev. D **78** (2008) 034006 [arXiv:0805.4119 [hep-ph]].
- [167] M. Antonelli *et al.*, “Flavor Physics in the Quark Sector,” Phys. Rept. **494** (2010) 197 [arXiv:0907.5386 [hep-ph]].
- [168] T. Hurth and M. Nakao, “Radiative and Electroweak Penguin Decays of B Mesons,” Ann. Rev. Nucl. Part. Sci. **60** (2010) 645 [arXiv:1005.1224 [hep-ph]].
- [169] S. Bertolini, F. Borzumati and A. Masiero, “QCD Enhancement Of Radiative B Decays,” Phys. Rev. Lett. **59** (1987) 180.
- [170] N. G. Deshpande, P. Lo, J. Trampetic, G. Eilam and P. Singer, “ $B \rightarrow K^* \gamma$ and the Top Quark Mass,” Phys. Rev. Lett. **59** (1987) 183.
- [171] B. Grinstein, R. P. Springer and M. B. Wise, “Effective Hamiltonian For Weak Radiative B Meson Decay,” Phys. Lett. B **202** (1988) 138. B. Grinstein, R. P. Springer and M. B. Wise, “Strong Interaction Effects in Weak Radiative B Meson Decay,” Nucl. Phys. B **339** (1990) 269.
- [172] R. Grigjanis, P. J. O’Donnell, M. Sutherland and H. Navelet, “QCD Corrected Effective Lagrangian for $B \rightarrow s$ Processes,” Phys. Lett. B **213** (1988) 355 [Erratum-ibid. B **286** (1992) 413].
- [173] R. Grigjanis, P. J. O’Donnell, M. Sutherland and H. Navelet, “Regularization in Low-Energy Standard Model Calculations,” Phys. Lett. B **237** (1990) 252.
- [174] G. Cella, G. Curci, G. Ricciardi and A. Vicer , “QCD Corrections To The Weak Radiative \bar{B} Meson Decay,” Phys. Lett. B **248**, 181 (1990).
- [175] G. Curci and G. Ricciardi, “QCD nonleading corrections to weak decays in the dimensional scheme,” Phys. Rev. D **47** (1993) 2991 [arXiv:hep-ph/9210215].
- [176] M. Misiak, “QCD Corrected Effective Hamiltonian For The $B \rightarrow X_s \gamma$ Decay,” Phys. Lett. B **269** (1991) 161.
- [177] M. Misiak, “The $B \rightarrow X_s e^+ e^-$ and $B \rightarrow X_s \gamma$ Decays With Next-To-Leading Logarithmic QCD Corrections,” Nucl. Phys. B **393** (1993) 23 [Erratum-ibid. B **439** (1995) 461].

- [178] P. L. Cho and B. Grinstein, “QCD enhancement of $b \rightarrow s\gamma$ decay for a heavy top quark,” Nucl. Phys. B **365** (1991) 279 [Erratum-ibid. B **427** (1994) 697].
- [179] R. Grigjanis, P. J. O’Donnell, M. Sutherland and H. Navelet, “QCD Corrections To $B \rightarrow s$ Processes: An Effective Lagrangian Approach,” Phys. Rept. **228** (1993) 93.
- [180] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, “Scheme independence of the effective Hamiltonian for $b \rightarrow s\gamma$ and $b \rightarrow sg$ decays,” Phys. Lett. B **316** (1993) 127 [arXiv:hep-ph/9307364].
- [181] M. Ciuchini, E. Franco, L. Reina and L. Silvestrini, “Leading order QCD corrections to $b \rightarrow s\gamma$ and $b \rightarrow sg$ decays in three regularization schemes,” Nucl. Phys. B **421** (1994) 41 [arXiv:hep-ph/9311357].
- [182] G. Cella, G. Curci, G. Ricciardi and A. Vicere, “QCD corrections to electroweak processes in an unconventional scheme: Application to the $b \rightarrow s\gamma$ decay,” Nucl. Phys. B **431** (1994) 417 [arXiv:hep-ph/9406203].
- [183] G. Cella, G. Curci, G. Ricciardi and A. Vicere, “The $b \rightarrow s\gamma$ decay revisited,” Phys. Lett. B **325** (1994) 227 [arXiv:hep-ph/9401254].
- [184] M. Misiak, “On The dimensional methods in rare b decays,” Phys. Lett. B **321** (1994) 113 [arXiv:hep-ph/9309236].
- [185] A. J. Buras, M. Misiak, M. Münz and S. Pokorski, “Theoretical Uncertainties And Phenomenological Aspects of $B \rightarrow X_s\gamma$ Decay,” Nucl. Phys. B **424** (1994) 374 [arXiv:hep-ph/9311345].
- [186] A. Ali, C. Greub and T. Mannel, “Rare B decays in the Standard Model,” Proc. of ECFA Workshop on the Physics of a B Meson Factory, Eds. R. Aleksan, A. Ali, 1993.
- [187] K. Adel and Y. P. Yao, “Effective Lagrangian for $b \rightarrow s$ processes with QCD corrections,” Mod. Phys. Lett. A **8** (1993) 1679 [arXiv:hep-ph/9302244].
- [188] K. Adel and Y. P. Yao, “Exact α_s calculation of $b \rightarrow s\gamma$ and $b \rightarrow s + g$,” Phys. Rev. D **49** (1994) 4945 [arXiv:hep-ph/9308349].
- [189] K. Adel and Y. P. Yao, “Renormalization of four quark operators, effective theory and the role of evanescent operators,” Phys. Rev. D **53** (1996) 374 [arXiv:hep-ph/9408341].
- [190] C. Greub and T. Hurth, “Two-loop matching of the dipole operators for $b \rightarrow s\gamma$ and $b \rightarrow s + g$,” Phys. Rev. D **56** (1997) 2934 [arXiv:hep-ph/9703349].
- [191] A. J. Buras, A. Kwiatkowski and N. Pott, “Scale uncertainties in the $B \rightarrow X_s\gamma$ decay,” Phys. Lett. B **414** (1997) 157 [Erratum-ibid. B **434** (1998) 459] [arXiv:hep-ph/9707482].

- [192] A. J. Buras, A. Kwiatkowski and N. Pott, “Next-to-leading order matching for the magnetic photon penguin operator in the $B \rightarrow X_s \gamma$ decay,” Nucl. Phys. B **517** (1998) 353 [arXiv:hep-ph/9710336].
- [193] M. Misiak and M. Münz, “Two loop mixing of dimension five flavor changing operators,” Phys. Lett. B **344** (1995) 308 [arXiv:hep-ph/9409454].
- [194] P. Gambino, M. Gorbahn and U. Haisch, “Anomalous dimension matrix for radiative and rare semileptonic B decays up to three loops,” Nucl. Phys. B **673** (2003) 238 [arXiv:hep-ph/0306079].
- [195] M. Czakon, U. Haisch and M. Misiak, “Four-loop anomalous dimensions for radiative flavour-changing decays,” JHEP **0703** (2007) 008 [arXiv:hep-ph/0612329].
- [196] K. G. Chetyrkin, M. Misiak and M. Münz, “Beta functions and anomalous dimensions up to three loops,” Nucl. Phys. B **518** (1998) 473 [arXiv:hep-ph/9711266].
- [197] K. G. Chetyrkin, M. Misiak and M. Münz, “Weak radiative B-meson decay beyond leading logarithms,” Phys. Lett. B **400** (1997) 206 [Erratum-ibid. B **425** (1998) 414] [arXiv:hep-ph/9612313].
- [198] M. Misiak and M. Steinhauser, “Three-loop matching of the dipole operators for $b \rightarrow s \gamma$ and $b \rightarrow s$ gluon,” Nucl. Phys. B **683** (2004) 277 [arXiv:hep-ph/0401041].
- [199] M. Gorbahn, U. Haisch and M. Misiak, “Three-loop mixing of dipole operators,” Phys. Rev. Lett. **95** (2005) 102004 [arXiv:hep-ph/0504194].
- [200] A. Ali and C. Greub, “Photon energy spectrum in $B \rightarrow X_s \gamma$ and comparison with data,” Phys. Lett. B **361** (1995) 146 [arXiv:hep-ph/9506374]. A. Ali and C. Greub, “Inclusive Photon Energy Spectrum In Rare B Decays,” Z. Phys. C **49** (1991) 431. A. Ali and C. Greub, “Rare decays $B \rightarrow X_d \gamma$ in the standard model,” Phys. Lett. B **287** (1992) 191.
- [201] N. Pott, “Bremsstrahlung corrections to the decay $b \rightarrow s \gamma$,” Phys. Rev. D **54** (1996) 938 [arXiv:hep-ph/9512252].
- [202] C. Greub, T. Hurth and D. Wyler, “Virtual corrections to the decay $b \rightarrow s \gamma$,” Phys. Lett. B **380** (1996) 385 [arXiv:hep-ph/9602281].
- [203] C. Greub, T. Hurth and D. Wyler, “Virtual $\mathcal{O}(\alpha_s)$ corrections to the inclusive decay $b \rightarrow s \gamma$,” Phys. Rev. D **54** (1996) 3350 [arXiv:hep-ph/9603404].
- [204] A. J. Buras, A. Czarnecki, M. Misiak and J. Urban, “Two-loop matrix element of the current-current operator in the decay $B \rightarrow X_s \gamma$,” Nucl. Phys. B **611** (2001) 488 [arXiv:hep-ph/0105160].

- [205] A. J. Buras, A. Czarnecki, M. Misiak and J. Urban, “Completing the NLO QCD calculation of $B \rightarrow X_s \gamma$,” Nucl. Phys. B **631** (2002) 219 [arXiv:hep-ph/0203135].
- [206] A. J. Buras and M. Misiak, “ $B \rightarrow X_s \gamma$ after completion of the NLO QCD calculations,” Acta Phys. Polon. B **33** (2002) 2597 [arXiv:hep-ph/0207131].
- [207] M. Kaminski, M. Misiak and M. Poradzinski, “Tree-level contributions to $B \rightarrow X_s \gamma$,” arXiv:1209.0965 [hep-ph].
- [208] P. Gambino and M. Misiak, “Quark mass effects in $B \rightarrow X_s \gamma$,” Nucl. Phys. B **611** (2001) 338 [arXiv:hep-ph/0104034].
- [209] M. Misiak, “NNLO QCD Calculations Of Rare B Decays,” Acta Phys. Polon. B **34** (2003) 4397.
- [210] H. M. Asatrian, C. Greub, A. Hovhannisyan, T. Hurth and V. Poghosyan, “Reduction of charm quark mass scheme dependence in $B \rightarrow X_s \gamma$ at the NNLL level,” Phys. Lett. B **619** (2005) 322 [arXiv:hep-ph/0505068].
- [211] K. Melnikov and A. Mitov, “The photon energy spectrum in $B \rightarrow X_s \gamma$ in perturbative QCD through $\mathcal{O}(\alpha_s^2)$,” Phys. Lett. B **620**, 69–79 (2005) [hep-ph/0505097].
- [212] I. R. Blokland, A. Czarnecki, M. Misiak, M. Slusarczyk and F. Tkachov, “The electromagnetic dipole operator effect on $B \rightarrow X_s \gamma$ at $\mathcal{O}(\alpha_s^2)$,” Phys. Rev. D **72** (2005) 033014 [arXiv:hep-ph/0506055].
- [213] H. M. Asatrian, A. Hovhannisyan, V. Poghosyan, T. Ewerth, C. Greub and T. Hurth, “NNLL QCD contribution of the electromagnetic dipole operator to $\Gamma(B \rightarrow X_s \gamma)$,” Nucl. Phys. B **749** (2006) 325 [arXiv:hep-ph/0605009].
- [214] H. M. Asatrian, T. Ewerth, A. Ferroglia, P. Gambino and C. Greub, “Magnetic dipole operator contributions to the photon energy spectrum in $B \rightarrow X_s \gamma$ at $\mathcal{O}(\alpha_s^2)$,” Nucl. Phys. B **762** (2007) 212 [arXiv:hep-ph/0607316].
- [215] H. M. Asatrian, T. Ewerth, H. Gabrielyan and C. Greub, “Charm quark mass dependence of the electromagnetic dipole operator contribution to $B \rightarrow X_s \gamma$ at $\mathcal{O}(\alpha_s^2)$,” Phys. Lett. B **647** (2007) 173 [arXiv:hep-ph/0611123].
- [216] H. M. Asatrian, T. Ewerth, A. Ferroglia, C. Greub and G. Ossola, “Complete (O_7, O_8) contribution to $B \rightarrow X_s \gamma$ at order α_s^2 ,” Phys. Rev. D **82** (2010) 074006 [arXiv:1005.5587 [hep-ph]].
- [217] T. Ewerth, “Fermionic corrections to the interference of the electro- and chromomagnetic dipole operators in $\bar{B} \rightarrow X_s \gamma$ at $\mathcal{O}(\alpha_s^2)$,” Phys. Lett. B **669**, 167–172 (2008) [arXiv:0805.3911].

- [218] Z. Ligeti, M.E. Luke, A.V. Manohar and M.B. Wise, “The $\bar{B} \rightarrow X_s \gamma$ photon spectrum,” Phys. Rev. D **60**, 034019 (1999) [hep-ph/9903305].
- [219] A. Ferroglia and U. Haisch, “Chromomagnetic Dipole-Operator Corrections in $\bar{B} \rightarrow X_s \gamma$ at $\mathcal{O}(\beta_0 \alpha_s^2)$,” Phys. Rev. D **82** (2010) 094012 [arXiv:1009.2144 [hep-ph]].
- [220] M. Misiak and M. Poradziński, “Completing the Calculation of BLM corrections to $B \rightarrow X_s \gamma$,” arXiv:1009.5685.
- [221] K. Bieri, C. Greub and M. Steinhauser, “Fermionic NNLL corrections to $b \rightarrow s \gamma$,” Phys. Rev. D **67** (2003) 114019 [arXiv:hep-ph/0302051].
- [222] M. Misiak and M. Steinhauser, “NNLO QCD corrections to the $B \rightarrow X_s \gamma$ matrix elements using interpolation in m_c ,” Nucl. Phys. B **764** (2007) 62 [arXiv:hep-ph/0609241].
- [223] M. Misiak and M. Steinhauser, “Large- m_c Asymptotic Behaviour of $\mathcal{O}(\alpha_s^2)$ Corrections to $B \rightarrow X_s \gamma$,” Nucl. Phys. B **840** (2010) 271 [arXiv:1005.1173 [hep-ph]].
- [224] R. Boughezal, M. Czakon and T. Schutzmeier, “NNLO fermionic corrections to the charm quark mass dependent matrix elements in $B \rightarrow X_s \gamma$,” JHEP **0709**, 072 (2007) [arXiv:0707.3090].
- [225] S. W. Bosch and G. Buchalla, “The radiative decays $B \rightarrow V \gamma$ at next-to-leading order in QCD,” Nucl. Phys. B **621**, 459 (2002) [hep-ph/0106081].
- [226] M. Beneke, T. Feldmann and D. Seidel, “Exclusive radiative and electroweak $b \rightarrow d$ and $b \rightarrow s$ penguin decays at NLO,” Eur. Phys. J. C **41** (2005) 173 [arXiv:hep-ph/0412400].
- [227] A. Ali, B. D. Pecjak and C. Greub, “ $B \rightarrow V \gamma$ Decays at NNLO in SCET,” Eur. Phys. J. C **55** (2008) 577 [arXiv:0709.4422 [hep-ph]].
- [228] M. Misiak, H.M. Asatrian, K. Bieri, M. Czakon, A. Czarnecki, T. Ewert, A. Ferroglia, P. Gambino, M. Gorbahn, C. Greub, U. Haisch, A. Hovhannisyan, T. Hurth, A. Mitov, V. Poghosyan, M. Ślusarczyk and M. Steinhauser, “Estimate of $B \rightarrow X_s \gamma$ at $\mathcal{O}(\alpha_s^2)$,” Phys. Rev. Lett. **98** (2007) 022002 [arXiv:hep-ph/0609232].
- [229] M. Misiak, “QCD challenges in radiative B decays,” AIP Conf. Proc. **1317** (2011) 276 [arXiv:1010.4896 [hep-ph]]. M. Misiak, “ $B \rightarrow X_s \gamma$ - Current Status,” Acta Phys. Polon. B **40** (2009) 2987 [arXiv:0911.1651 [hep-ph]]. M. Misiak, “QCD Calculations of Radiative B Decays,” arXiv:0808.3134 [hep-ph].
- [230] U. Haisch, “ $\bar{B} \rightarrow X_s \gamma$: Standard Model and Beyond,” arXiv:0805.2141 [hep-ph]. U. Haisch, “Recent Developments in $\bar{B} \rightarrow X_s \gamma$,” arXiv:0706.2056 [hep-ph].

- [231] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, “On The Elimination Of Scale Ambiguities In Perturbative Quantum Chromodynamics,” *Phys. Rev. D* **28**, 228–234 (1983).
- [232] T. Schutzmeier, Matrix elements for the $\bar{B} \rightarrow X_s \gamma$ decay at NNLO, Ph.D. Thesis, University of Würzburg, 2009.
- [233] R. Boughezal, M. Czakon and T. Schutzmeier, in preparation.
- [234] M. Czakon, P. Fiedler, T. Huber and T. Schutzmeier, in progress.
- [235] M. Czakon, R. N. Lee, M. Misiak, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, in progress.
- [236] C. Greub and P. Liniger, “The rare decay $b \rightarrow s g$ beyond leading logarithms,” *Phys. Lett. B* **494** (2000) 237 [arXiv:hep-ph/0008071].
- [237] C. Greub and P. Liniger, “Calculation of next-to-leading QCD corrections to $b \rightarrow s g$,” *Phys. Rev. D* **63** (2001) 054025 [arXiv:hep-ph/0009144].
- [238] A. Ali, H. Asatrian and C. Greub, “Inclusive decay rate for $B \rightarrow X_d + \gamma$ in next-to-leading logarithmic order and CP asymmetry in the standard model,” *Phys. Lett. B* **429** (1998) 87 [arXiv:hep-ph/9803314].
- [239] T. Hurth, E. Lunghi and W. Porod, “Untagged $B \rightarrow X_{s,d} \gamma$ CP asymmetry as a probe for new physics,” *Nucl. Phys. B* **704**, 56 (2005) [arXiv:hep-ph/0312260].
- [240] T. Hurth, E. Lunghi and W. Porod, “Updated NLL results for $B \rightarrow X_{s,d} \gamma$ in and beyond the SM,” *Eur. Phys. J. C* **33** (2004) S382 [arXiv:hep-ph/0310282].
- [241] M. Beneke, T. Feldmann and D. Seidel, “Systematic approach to exclusive $B \rightarrow V l^+ l^-, V \gamma$ decays,” *Nucl. Phys. B* **612**, 25 (2001) [hep-ph/0106067].
- [242] A. Ali, E. Lunghi and A. Y. Parkhomenko, “Implication of the $B \rightarrow (\rho, \omega) \gamma$ branching ratios for the CKM phenomenology,” *Phys. Lett. B* **595**, 323 (2004) [hep-ph/0405075].
- [243] S. W. Bosch and G. Buchalla, “Constraining the unitarity triangle with $B \rightarrow V \gamma$,” *JHEP* **0501** (2005) 035 [arXiv:hep-ph/0408231].
- [244] T. Becher, R. J. Hill and M. Neubert, “Factorization in $B \rightarrow V \gamma$ decays,” *Phys. Rev. D* **72** (2005) 094017 [arXiv:hep-ph/0503263].
- [245] P. Ball, G. W. Jones and R. Zwicky, “ $B \rightarrow V \gamma$ beyond QCD factorisation,” *Phys. Rev. D* **75**, 054004 (2007) [arXiv:hep-ph/0612081].

- [246] F. J. Gilman and M. B. Wise, “ $K \rightarrow \pi e^+ e^-$ In The Six Quark Model,” Phys. Rev. D **21** (1980) 3150. G. Ecker, A. Pich and E. de Rafael, “Radiative Kaon Decays and CP Violation in Chiral Perturbation Theory,” Nucl. Phys. B **303** (1988) 665.
- [247] G. Buchalla, G. D’Ambrosio and G. Isidori, “Extracting short-distance physics from $K_{L,S} \rightarrow \pi^0 e^+ e^-$ decays,” Nucl. Phys. B **672** (2003) 387 [arXiv:hep-ph/0308008].
- [248] G. Isidori, C. Smith and R. Unterdorfer, “The rare decay $K_{L,S} \rightarrow \pi^0 \mu^+ \mu^-$ within the SM,” Eur. Phys. J. C **36** (2004) 57 [arXiv:hep-ph/0404127].
- [249] S. Friot, D. Greynat and E. De Rafael, “Rare kaon decays revisited,” Phys. Lett. B **595** (2004) 301 [arXiv:hep-ph/0404136].
- [250] F. Mescia, C. Smith and S. Trine, “ $K_{L,S} \rightarrow \pi^0 e^+ e^-$ and $K_L \rightarrow \pi^0 \mu^+ \mu^-$: A binary star on the stage of flavor physics,” JHEP **0608** (2006) 088 [arXiv:hep-ph/0606081].
- [251] C. Dib, I. Dunietz and F. J. Gilman, “CP violation in the $K_L \rightarrow \pi^0 l^+ l^-$ Decay Amplitude for large m_t ,” Phys. Lett. B **218** (1989) 487. C. Dib, I. Dunietz and F. J. Gilman, “ $K_L \rightarrow \pi^0 l^+ l^-$ for Large m_t ,” Phys. Rev. D **39** (1989) 2639.
- [252] J. Flynn and L. Randall, “The CP Violating Contribution to the decay $K_L \rightarrow \pi^0 l^+ l^-$,” Nucl. Phys. B **326** (1989) 31 [Erratum-ibid. B **334** (1990) 580].
- [253] A. J. Buras, M. E. Lautenbacher, M. Misiak and M. Münz, “Direct CP Violation in $K_L \rightarrow \pi^0 e^+ e^-$ Beyond Leading Logarithms,” Nucl. Phys. B **423** (1994) 349 [arXiv:hep-ph/9402347].
- [254] A. J. Buras and M. Münz, “Effective Hamiltonian for $B \rightarrow X_s e^+ e^-$ beyond leading logarithms in the NDR and HV schemes,” Phys. Rev. D **52** (1995) 186 [arXiv:hep-ph/9501281].
- [255] G. Burdman, “Short distance coefficients and the vanishing of the lepton asymmetry in $B \rightarrow V \ell^+ \ell^-$ lepton-,” Phys. Rev. D **57** (1998) 4254 [arXiv:hep-ph/9710550].
- [256] H. H. Asatryan, H. M. Asatryan, C. Greub and M. Walker, “Calculation of two loop virtual corrections to $b \rightarrow s l^+ l^-$ in the standard model,” Phys. Rev. D **65** (2002) 074004 [arXiv:hep-ph/0109140]. H. H. Asatryan, H. M. Asatryan, C. Greub and M. Walker, “Two-loop virtual corrections to $B \rightarrow X_s l^+ l^-$ in the standard model,” Phys. Lett. B **507** (2001) 162 [arXiv:hep-ph/0103087].
- [257] H. H. Asatryan, H. M. Asatryan, C. Greub and M. Walker, “Complete gluon bremsstrahlung corrections to the process $b \rightarrow s l^+ l^-$,” Phys. Rev. D **66** (2002) 034009 [arXiv:hep-ph/0204341].

- [258] A. Ghinculov, T. Hurth, G. Isidori and Y. P. Yao, “The rare decay $B \rightarrow X_s l^+ l^-$ to NNLL precision for arbitrary dilepton invariant mass,” Nucl. Phys. B **685** (2004) 351 [arXiv:hep-ph/0312128]. A. Ghinculov, T. Hurth, G. Isidori and Y. P. Yao, “New NNLL QCD results on the decay $B \rightarrow X_s l^+ l^-$,” Eur. Phys. J. C **33** (2004) S288 [arXiv:hep-ph/0310187].
- [259] A. Ghinculov, T. Hurth, G. Isidori and Y. P. Yao, “Forward-backward asymmetry in $B \rightarrow X_s l^+ l^-$ at the NNLL level,” Nucl. Phys. B **648** (2003) 254 [arXiv:hep-ph/0208088].
- [260] H. M. Asatrian, K. Bieri, C. Greub and A. Hovhannisyan, “NNLL corrections to the angular distribution and to the forward-backward asymmetries in $B \rightarrow X_s l^+ l^-$,” Phys. Rev. D **66** (2002) 094013 [arXiv:hep-ph/0209006].
- [261] H. M. Asatrian, H. H. Asatryan, A. Hovhannisyan and V. Poghosyan, “Complete bremsstrahlung corrections to the forward-backward asymmetries in $b \rightarrow X_s l^+ l^-$,” Mod. Phys. Lett. A **19** (2004) 603 [arXiv:hep-ph/0311187].
- [262] C. Greub, V. Pilipp and C. Schupbach, “Analytic calculation of two-loop QCD corrections to $b \rightarrow s \ell^+ \ell^-$ in the high q^2 region,” JHEP **0812** (2008) 040 [arXiv:0810.4077 [hep-ph]].
- [263] C. Bobeth, P. Gambino, M. Gorbahn and U. Haisch, “Complete NNLO QCD analysis of $\bar{B} \rightarrow X_s l^+ l^-$ and higher order electroweak effects,” JHEP **0404** (2004) 071 [arXiv:hep-ph/0312090].
- [264] T. Huber, E. Lunghi, M. Misiak and D. Wyler, “Electromagnetic logarithms in $\bar{B} \rightarrow X_s l^+ l^-$,” Nucl. Phys. B **740** (2006) 105 [arXiv:hep-ph/0512066].
- [265] T. Huber, T. Hurth and E. Lunghi, “Logarithmically Enhanced Corrections to the Decay Rate and Forward Backward Asymmetry in $\bar{B} \rightarrow X_s \ell^+ \ell^-$,” Nucl. Phys. B **802** (2008) 40 [arXiv:0712.3009 [hep-ph]].
- [266] T. Huber, T. Hurth and E. Lunghi, “The Role of Collinear Photons in the Rare Decay $\bar{B} \rightarrow X_s \ell^+ \ell^-$,” arXiv:0807.1940 [hep-ph].
- [267] G. Bell, M. Beneke, T. Huber and X. Q. Li, “Heavy-to-light currents at NNLO in SCET and semi-inclusive $B \rightarrow X_s l^+ l^-$ ” Nucl. Phys. B **843** (2011) 143 [arXiv:1007.3758 [hep-ph]].
- [268] H. M. Asatrian, K. Bieri, C. Greub and M. Walker, “Virtual- and bremsstrahlung corrections to $b \rightarrow d l^+ l^-$ in the standard model,” Phys. Rev. D **69** (2004) 074007 [arXiv:hep-ph/0312063].
- [269] D. Seidel, “Analytic two-loop virtual corrections to $b \rightarrow d l^+ l^-$,” Phys. Rev. D **70** (2004) 094038 [arXiv:hep-ph/0403185].

- [270] J. Charles, A. Le Yaouanc, L. Oliver, O. Pene and J. C. Raynal, “Heavy-to-light form factors in the heavy mass to large energy limit of QCD,” *Phys. Rev. D* **60**, 014001 (1999) [hep-ph/9812358].
- [271] A. Ali, P. Ball, L. T. Handoko and G. Hiller, “A comparative study of the decays $B \rightarrow (K, K^*)l^+l^-$ in standard model and supersymmetric theories,” *Phys. Rev. D* **61** (2000) 074024 [arXiv:hep-ph/9910221].
- [272] M. Beneke and T. Feldmann, “Symmetry-breaking corrections to heavy-to-light B meson form factors at large recoil,” *Nucl. Phys. B* **592**, 3 (2001) [hep-ph/0008255].
- [273] A. Ali, G. Kramer and G. H. Zhu, “ $B \rightarrow K^*l^+l^-$ in soft-collinear effective theory,” *Eur. Phys. J. C* **47** (2006) 625 [arXiv:hep-ph/0601034].
- [274] C. Bobeth, G. Hiller and G. Piranishvili, “CP Asymmetries in $\bar{B} \rightarrow \bar{K}^*(\rightarrow \bar{K}\pi)\bar{\ell}\ell$ and Untagged $\bar{B}_s, B_s \rightarrow \phi(\rightarrow K^+K^-)\bar{\ell}\ell$ Decays at NLO,” *JHEP* **0807** (2008) 106 [arXiv:0805.2525 [hep-ph]].
- [275] U. Egede, T. Hurth, J. Matias, M. Ramon and W. Reece, “New observables in the decay mode $\bar{B} \rightarrow \bar{K}^{*0}l^+l^-$,” *JHEP* **0811** (2008) 032 [arXiv:0807.2589 [hep-ph]].
- [276] W. Altmannshofer, P. Ball, A. Bharucha, A. J. Buras, D. M. Straub and M. Wick, “Symmetries and Asymmetries of $B \rightarrow K^*\mu^+\mu^-$ Decays in the Standard Model and Beyond,” *JHEP* **0901** (2009) 019 [arXiv:0811.1214 [hep-ph]].
- [277] A. Bharucha, T. Feldmann and M. Wick, “Theoretical and Phenomenological Constraints on Form Factors for Radiative and Semi-Leptonic B-Meson Decays,” *JHEP* **1009** (2010) 090 [arXiv:1004.3249 [hep-ph]].
- [278] M. Beylich, G. Buchalla and T. Feldmann, “Theory of $B \rightarrow K^{(*)}l^+l^-$ decays at high q^2 : OPE and quark-hadron duality,” *Eur. Phys. J. C* **71** (2011) 1635 [arXiv:1101.5118 [hep-ph]].
- [279] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, “QCD factorization for $B \rightarrow \pi\pi$ decays: Strong phases and CP violation in the heavy quark limit,” *Phys. Rev. Lett.* **83** (1999) 1914 [arXiv:hep-ph/9905312].
- [280] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, “QCD factorization for exclusive, non-leptonic B meson decays: General arguments and the case of heavy-light final states,” *Nucl. Phys. B* **591** (2000) 313 [arXiv:hep-ph/0006124].
- [281] J. Chay, “ $\bar{B} \rightarrow D^{(*)}\pi^-$ beyond naive factorization in the heavy quark limit,” *Phys. Lett. B* **476** (2000) 339 [arXiv:hep-ph/0001266].

- [282] H. D. Politzer and M. B. Wise, “Perturbative corrections to factorization in \bar{B} decay,” Phys. Lett. B **257** (1991) 399.
- [283] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, “QCD factorization in $B \rightarrow \pi K$, $\pi\pi$ decays and extraction of Wolfenstein parameters,” Nucl. Phys. B **606** (2001) 245 [arXiv:hep-ph/0104110].
- [284] M. Beneke and M. Neubert, “QCD factorization for $B \rightarrow PP$ and $B \rightarrow PV$ decays,” Nucl. Phys. B **675** (2003) 333 [arXiv:hep-ph/0308039].
- [285] M. Beneke, M. Neubert, “Flavor singlet B decay amplitudes in QCD factorization,” Nucl. Phys. **B651** (2003) 225-248. [hep-ph/0210085].
- [286] A. L. Kagan, “Polarization in $B \rightarrow VV$ decays,” Phys. Lett. B **601** (2004) 151 [arXiv:hep-ph/0405134].
- [287] M. Beneke, J. Rohrer and D. Yang, “Branching fractions, polarisation and asymmetries of $B \rightarrow VV$ decays,” Nucl. Phys. B **774** (2007) 64 [arXiv:hep-ph/0612290]; M. Beneke, J. Rohrer and D. Yang, “Enhanced electroweak penguin amplitude in $B \rightarrow VV$ decays,” Phys. Rev. Lett. **96** (2006) 141801 [arXiv:hep-ph/0512258].
- [288] H. Y. Cheng and K. C. Yang, “Branching Ratios and Polarization in $B \rightarrow VV$, VA , AA Decays,” Phys. Rev. D **78** (2008) 094001 [Erratum-ibid. D **79** (2009) 039903] [arXiv:0805.0329 [hep-ph]].
- [289] M. Bartsch, G. Buchalla and C. Kraus, “ $B \rightarrow V_L V_L$ Decays at Next-to-Leading Order in QCD,” arXiv:0810.0249 [hep-ph].
- [290] D. Fakirov and B. Stech, “F And D Decays,” Nucl. Phys. B **133** (1978) 315.
- [291] J. D. Bjorken, “Topics In B Physics,” Nucl. Phys. Proc. Suppl. **11** (1989) 325.
- [292] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, “An effective field theory for collinear and soft gluons: Heavy to light decays,” Phys. Rev. D **63** (2001) 114020 [arXiv:hep-ph/0011336].
- [293] C. W. Bauer, D. Pirjol and I. W. Stewart, “Soft-collinear factorization in effective field theory,” Phys. Rev. D **65** (2002) 054022 [arXiv:hep-ph/0109045].
- [294] C. W. Bauer, D. Pirjol and I. W. Stewart, “A proof of factorization for $B \rightarrow D\pi$,” Phys. Rev. Lett. **87** (2001) 201806 [arXiv:hep-ph/0107002].
- [295] M. Beneke and S. Jäger, “Spectator scattering at NLO in non-leptonic B decays: Tree amplitudes,” Nucl. Phys. B **751** (2006) 160 [arXiv:hep-ph/0512351].

- [296] N. Kivel, “Radiative corrections to hard spectator scattering in $B \rightarrow \pi\pi$ decays,” JHEP **0705** (2007) 019 [arXiv:hep-ph/0608291].
- [297] M. Beneke and S. Jäger, “Spectator scattering at NLO in non-leptonic B decays: Leading penguin amplitudes,” Nucl. Phys. B **768** (2007) 51 [arXiv:hep-ph/0610322].
- [298] A. Jain, I. Z. Rothstein and I. W. Stewart, “Penguin Loops for Nonleptonic B-Decays in the Standard Model: Is there a Penguin Puzzle?,” arXiv:0706.3399 [hep-ph].
- [299] V. Pilipp, “Hard spectator interactions in $B \rightarrow \pi\pi$ at order α_s^2 ,” Nucl. Phys. B **794** (2008) 154 [arXiv:0709.3214 [hep-ph]].
- [300] G. Bell, “NNLO Vertex Corrections in charmless hadronic B decays: Imaginary part,” Nucl. Phys. B **795** (2008) 1 [arXiv:0705.3127 [hep-ph]].
- [301] G. Bell, “NNLO vertex corrections in charmless hadronic B decays: Real part,” Nucl. Phys. B **822** (2009) 172 [arXiv:0902.1915 [hep-ph]].
- [302] G. Bell and V. Pilipp, “ $B \rightarrow \pi^-\pi^0/\rho^-\rho^0$ to NNLO in QCD factorization,” Phys. Rev. D **80** (2009) 054024 [arXiv:0907.1016 [hep-ph]].
- [303] M. Beneke, T. Huber and X. Q. Li, “NNLO vertex corrections to non-leptonic B decays: Tree amplitudes,” Nucl. Phys. B **832** (2010) 109 [arXiv:0911.3655 [hep-ph]].
- [304] R. Fleischer, N. Serra and N. Tuning, “Tests of Factorization and SU(3) Relations in B Decays into Heavy-Light Final States,” Phys. Rev. D **83** (2011) 014017 [arXiv:1012.2784 [hep-ph]].
- [305] H. Y. Cheng and K. C. Yang, “Charmless $B \rightarrow VV$ decays in QCD factorization: Implications of recent $B \rightarrow \phi K^*$ measurement,” Phys. Lett. B **511** (2001) 40 [arXiv:hep-ph/0104090].
- [306] A. Ali and A. Y. Parkhomenko, “Branching ratios for $B \rightarrow \rho\gamma$ decays in next-to-leading order in α_s including hard spectator corrections,” Eur. Phys. J. C **23** (2002) 89 [arXiv:hep-ph/0105302].
- [307] S. Descotes-Genon and C. T. Sachrajda, “Spectator interactions in $B \rightarrow V\gamma$ decays and QCD factorisation,” Nucl. Phys. B **693** (2004) 103 [arXiv:hep-ph/0403277].
- [308] J. g. Chay and C. Kim, “Rare radiative exclusive B decays in soft-collinear effective theory,” Phys. Rev. D **68** (2003) 034013 [arXiv:hep-ph/0305033].
- [309] R. Fleischer, “Flavour Physics and CP Violation: Expecting the LHC,” arXiv:0802.2882 [hep-ph].

- [310] J. Rosiek, P. Chankowski, A. Dedes, S. Jäger and P. Tanedo, “SUSY-FLAVOR: a computational tool for FCNC and CP-violating processes in the MSSM,” *Comput. Phys. Commun.* **181** (2010) 2180 [arXiv:1003.4260 [hep-ph]].
- [311] H. Burkhardt et al., *Phys. Lett.* **B206**, (1988) 169; G. D. Barr *et al.* [NA31 Collaboration], “A New measurement of direct CP violation in the neutral kaon system,” *Phys. Lett.* **B317** (1993) 233.
- [312] L. K. Gibbons *et al.*, “Measurement of the CP violation parameter $\text{Re}(\epsilon'/\epsilon)$,” *Phys. Rev. Lett.* **70** (1993) 1203.
- [313] C. Lehner and C. Sturm, “Matching factors for Delta S=1 four-quark operators in RI/SMOM schemes,” *Phys. Rev. D* **84** (2011) 014001 [arXiv:1104.4948 [hep-ph]].
- [314] V. Lubicz and C. Tarantino, “Flavour physics and Lattice QCD: averages of lattice inputs for the Unitarity Triangle Analysis,” *Nuovo Cim. B* **123** (2008) 674 [arXiv:0807.4605 [hep-lat]].
- [315] P. Dimopoulos, “ $K^0 - \bar{K}^0$ on the Lattice,” arXiv:1101.3069 [hep-lat].
- [316] J. Shigemitsu, “Progress in Lattice QCD Relevant for Flavor Physics,” arXiv:1102.0716 [hep-ph].
- [317] Y. Aoki, R. Arthur, T. Blum, P. A. Boyle, D. Brommel, N. H. Christ, C. Dawson and T. Izubuchi *et al.*, “Continuum Limit of B_K from 2+1 Flavor Domain Wall QCD,” *Phys. Rev. D* **84** (2011) 014503 [arXiv:1012.4178 [hep-lat]].
- [318] B. D. Gaiser, T. Tsao and M. B. Wise, “Parameters Of The Six Quark Model,” *Annals Phys.* **132** (1981) 66.
- [319] A. J. Buras and J. M. Gérard, “ $1/N$ Expansion For Kaons,” *Nucl. Phys. B* **264** (1986) 371.
- [320] J. M. Gérard, “Electroweak interactions of hadrons,” *Acta Phys. Polon. B* **21** (1990) 257.
- [321] J. M. Gérard, “An upper bound on the Kaon B-parameter and $\text{Re} \epsilon_K$,” arXiv:1012.2026 [hep-ph].
- [322] J. Bijnens and J. Prades, “The B_K parameter in the $1/N$ expansion,” *Nucl. Phys. B* **444** (1995) 523 [arXiv:hep-ph/9502363].
- [323] H. M. Asatrian, C. Greub, A. Kokulu and A. Yeghiazaryan, “NLL QCD contribution of the Electromagnetic Dipole operator to $\bar{B} \rightarrow X_s \gamma \gamma$,” *Phys. Rev. D* **85** (2012) 014020 [arXiv:1110.1251 [hep-ph]].

Table 7: Summary of NLO and NNLO summits conquered by the members of MNLC.

Decay	NLO	NNLO
Current-Current (Q_1, Q_2) QCD P (Q_3, Q_4, Q_5, Q_6) EW P (Q_7, Q_8, Q_9, Q_{10}) $Br(B)_{SL}$ inclusive non-leptonic decays B-Decays in QCDF Current-Current (BSM) Penguins (BSM) Semi-Leptonic ($ V_{cb} , V_{ub} $)	Buras, Weisz Buras,Jamin,Lautenbacher,Weisz Fleischer Buras, Jamin, Lautenbacher Ball, Lenz, Nierste, Ostermaier Ball, Buchalla, Jamin Bartsch, Buchalla Buras, Misiak, Urban Buras, Misiak, Urban Ball, Czarnecki, Gambino	Gorbahn, Haisch Gorbahn, Haisch Buras, Gambino, Haisch Bell, Pilipp Czarnecki, Gambino
η_1 η_2, η_B η_3 Tree-Level FCNC $\Delta\Gamma_{B_s}, \Delta\Gamma_{B_d}$	Herrlich, Nierste Buras, Jamin, Weisz; Urban Herrlich, Nierste Buras, Girrbach Buchalla, Lenz, Nierste	Brod, Gorbahn Brod, Gorbahn
$K_L^0 \rightarrow \pi^0 \nu \bar{\nu}, B \rightarrow l^+ l^-, X_s \nu \bar{\nu}$ $K^+ \rightarrow \pi^+ \nu \bar{\nu}, K_L \rightarrow \mu^+ \mu^-$ $K^+ \rightarrow \pi^+ \mu \bar{\mu}$ EW to Charm in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ EW to Top in $K \rightarrow \pi \nu \bar{\nu}$	Buchalla, Buras ; Misiak, Urban Buchalla, Buras Buchalla, Buras Brod, Gorbahn Buchalla, Buras; Brod, Gorbahn Stamou	Buras, Gorbahn, Haisch Nierste
$B \rightarrow X_s \gamma$		
$C_i(M_W)$ $\hat{\gamma}(Q_{7\gamma}, Q_{8G})$ $\hat{\gamma}(\text{Mixing})$ Matrix Elements $B \rightarrow K^*(\rho) \gamma$	Buras, Kwiatkowski, Pott; Gambino Misiak, Münz; Chetyrkin, Misiak, Münz Buras, Czarnecki, Misiak, Pott, Urban Bosch, Buchalla	Bobeth, Misiak, Urban Gorbahn, Haisch, Misiak Haisch, Misiak Ewerth; Haisch; Misiak
$K_L \rightarrow \pi^0 e^+ e^-$ $B \rightarrow X_s \ell^+ \ell^-: C_i(M_W)$ $\hat{\gamma}(\text{Mixing})$ Matrix Elements	Buras, Lautenbacher, Misiak, Münz Misiak; Buras, Münz Buras, Lautenbacher, Misiak, Münz Misiak; Buras, Münz	Bobeth, Misiak, Urban Gambino, Gorbahn, Haisch Bobeth, Gambino, Haisch Bell, Pilipp

Table 8: Summary of all NLO and NNLO Calculations for Weak Decays

Decay	NLO	NNLO
Current-Current (Q_1, Q_2)	[5, 90]	[93]
QCD penguins (Q_3, Q_4, Q_5, Q_6)	[94, 95, 96, 97, 98, 99]; [100]	[93]
electroweak penguins (Q_7, Q_8, Q_9, Q_{10})	[101, 96, 97, 98], [100]	[53]
$Br(B)_{SL}$	[102, 103, 104, 105]	
inclusive $\Delta S = 1$ decays	[5, 6, 106, 107, 108]; [109]	
Two-Body B-Decays in QCDF	[279]–[289]	[295]–[303]
Current-Current (BSM)	[30, 29]	
Penguins (BSM)	[29]	
Semi-Leptonic ($ V_{cb} , V_{ub} $) B Decays	[121]–[126], [134, 135]	[126]–[133], [136]–[139]
η_1	[143]	[147]
η_2, η_B	[142, 64]	
η_3	[144, 145]	[146]
Tree-Level FCNC	[156, 157]	
ADMs BSM	[30, 29]	
$\Delta\Gamma_{B_s}$	[150, 151, 152, 154]	
$\Delta\Gamma_{B_d}$	[152, 153, 154]	
$K_L^0 \rightarrow \pi^0 \nu \bar{\nu}, B \rightarrow l^+ l^-, B \rightarrow X_s \nu \bar{\nu}$	[48, 49, 160, 161]	
$K^+ \rightarrow \pi^+ \nu \bar{\nu}, K_L \rightarrow \mu^+ \mu^-$	[51, 161]	[31, 52]
$K^+ \rightarrow \pi^+ \mu \bar{\mu}$	[165]	
EW to Charm in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$	[166]	
EW to Top in $K \rightarrow \pi \nu \bar{\nu}$	[56, 57]	
$B \rightarrow X_s \gamma: C_i(M_W)$	[188]–[190], [191, 192, 67]	[54, 55, 198]
$\hat{\gamma}(Q_{7\gamma}, Q_{8G})$	[193, 194]	[199]
$\hat{\gamma}(\text{Mixing})$	[197, 196, 194]	[195]
Matrix Elements	[200, 201, 202, 203, 204, 205]	[210]–[224]
$B \rightarrow K^*(\rho) \gamma$	[241, 225, 306]	[226, 227]
$K_L \rightarrow \pi^0 e^+ e^-$	[253]	
$B \rightarrow X_s \ell^+ \ell^-: C_i(M_W)$	[177, 254]	[54]
$\hat{\gamma}(\text{Mixing})$	[177, 254]	[194]
Matrix Elements	[177, 254]	[256]–[267]
$B \rightarrow X_d \ell^+ \ell^-$	[177, 254]	[268, 269]
$B \rightarrow K^* \mu^+ \mu^-$	[270, 271, 272]	[241, 226]